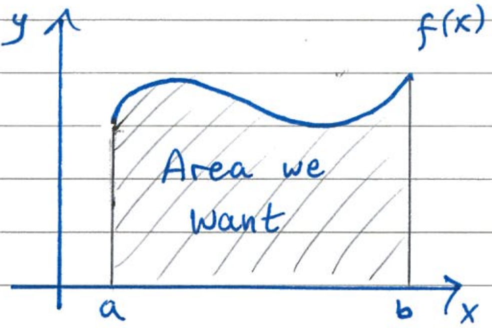
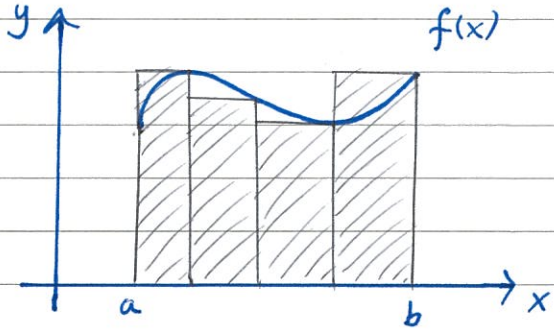
§8.3 - Area and the Definite Integral

In many applications, knowing the <u>area under a function</u> is important!





Often the area is not a nice geometric shape, so we may need to <u>approximate with rectangles/trapezoids.</u>

<u>Definition:</u> The <u>definite integral</u> of $f(x)$ from $x=a$ to $x=b$ is the area under $f(x)$ between a and b.

upper bound

$$\boxed{\int_a^b f(x)\,dx = \text{area under } f(x) \text{ from } x=a \text{ to } x=b.}$$

integrand   with respect to x

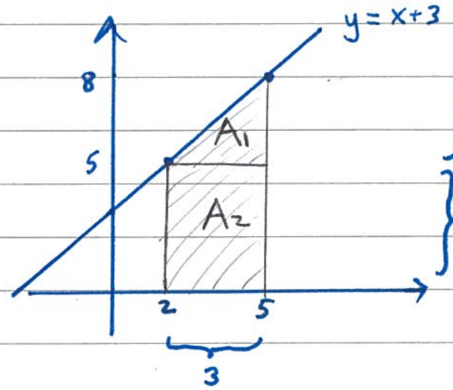
lower bound

If $f(x)$ is nice enough, we can calculate $\int_a^b f(x)\,dx$ using our geometry knowledge!

Ex: Calculate the following definite integrals using geometry.

1. $\int_2^5 x+3\, dx$

Solution:



A_1 = Area of triangle

$= \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$

A_2 = Area of rectangle

$= 3 \cdot 5 = 15$

So $\int_2^5 x+3\, dx$ = Total Area

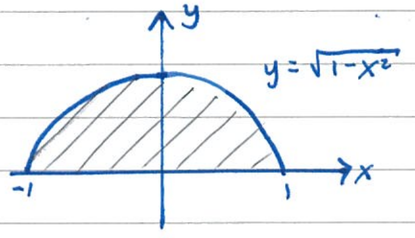
$= A_1 + A_2$

$= 9/2 + 15 = \boxed{39/2}$

2. $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

Solution: Well... if $y = \sqrt{1-x^2}$, then $y^2 = 1-x^2$, so $x^2+y^2 = 1$.

This is a circle of radius 1 centered at $(0,0)$ (but only top half... $y = \sqrt{1-x^2} \geq 0$)

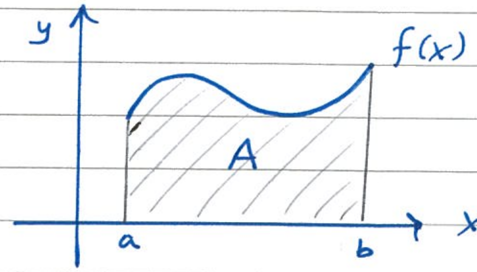


So $\int_{-1}^{1} \sqrt{1-x^2}\, dx = \dfrac{\text{Area of circle}}{2}$

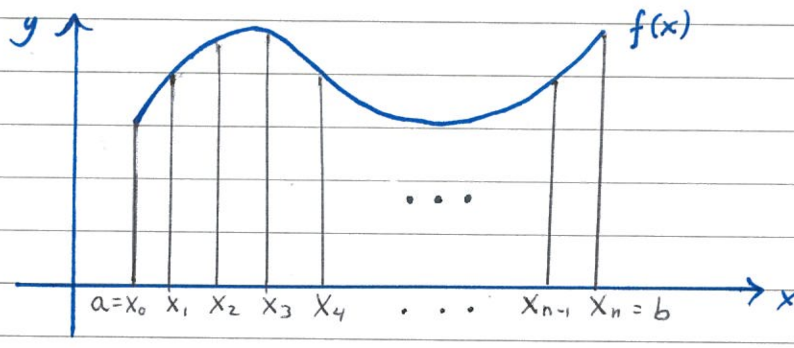
$= \frac{1}{2}(\pi r^2)$

$= \boxed{\dfrac{\pi}{2}}$ $(r = 1)$

How can we approximate the area when it's not so nice??



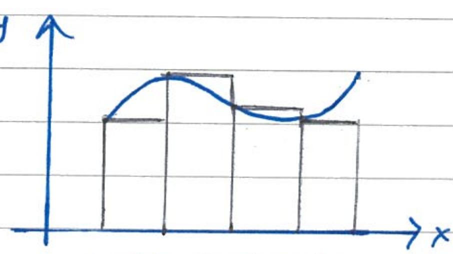
Start by dividing the interval [a, b] into n subintervals of equal width.



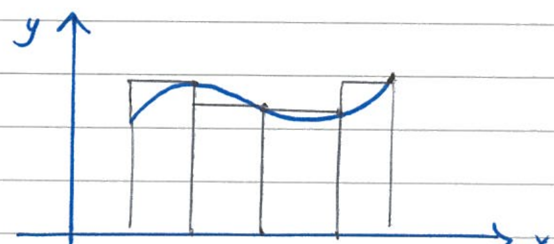
Let ΔX = width of each subinterval $= \dfrac{b-a}{n}$

So $x_i = x_0 + i \cdot \Delta X$

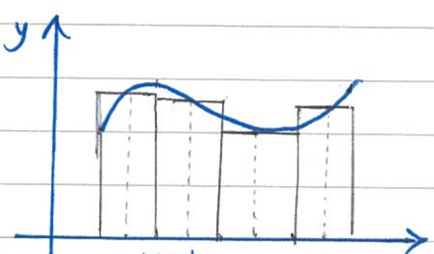
4 methods of approximation:



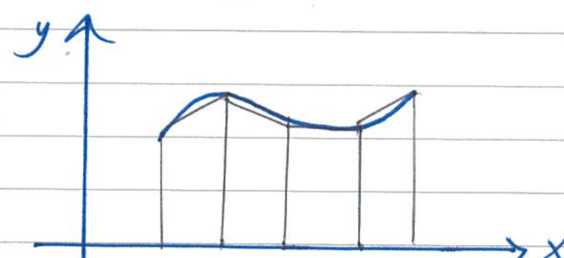
Left Endpoints



Right Endpoints



Midpoints



Trapezoids

What area do we get from each method?

Left Endpoints:

$$\text{Area} = \Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \cdots + \Delta x \cdot f(x_{n-1})$$

$$= \boxed{\Delta x \left[f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right]}$$

(all except x_n)

Right Endpoints:

$$\text{Area} = \boxed{\Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n) \right]}$$

(all except x_0)

Midpoints:

Say the midpoints are $m_1, m_2, \ldots, m_n$.

Then $\text{Area} = \boxed{\Delta x \left[f(m_1) + f(m_2) + \cdots + f(m_n) \right]}$

Trapezoids:

$$\text{Area} = \boxed{\Delta x \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right]}$$

(half the first and last, plus the rest)

Ex: Use 4 subintervals ($n=4$) to approximate $\int_0^4 x^2 + x \, dx$ using each method.

Solution:

First of all, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-0}{4} = 1$

$x_0 = 0$ $\quad x_1 = 1$ $\quad x_2 = 2$ $\quad x_3 = 3$ $\quad x_4 = 4$

$m_1 = 0.5$ $\quad m_2 = 1.5$ $\quad m_3 = 2.5$ $\quad m_4 = 3.5$

We have $f(0) = 0$ $\qquad f(0.5) = 3/4$

$f(1) = 2$ $\qquad f(1.5) = 15/4$

$f(2) = 6$ $\qquad f(2.5) = 35/4$

$f(3) = 12$ $\qquad f(3.5) = 63/4$

$f(4) = 20$

Left Endpoints: Area $\approx \Delta x \left[f(x_0) + f(x_1) + f(x_2) + f(x_3) \right]$
$= 1 \left[0 + 2 + 6 + 12 \right]$
$= \boxed{20}$

Right Endpoints: Area $\approx \Delta x \left[f(x_1) + f(x_2) + f(x_3) + f(x_4) \right]$
$= 1 \left[2 + 6 + 12 + 20 \right]$
$= \boxed{40}$

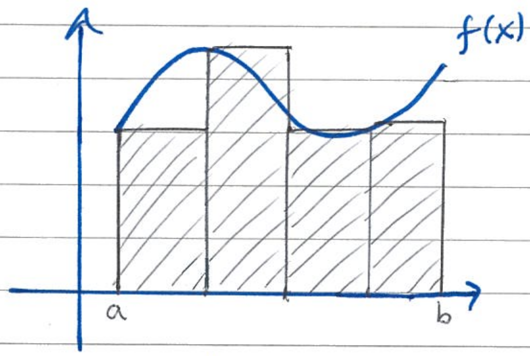
Midpoints: Area $\approx \Delta x \left[f(m_1) + f(m_2) + f(m_3) + f(m_4) \right]$
$= 1 \left[3/4 + 15/4 + 35/4 + 63/4 \right]$
$= \boxed{29}$

Trapezoids: Area $\approx \Delta x \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right]$

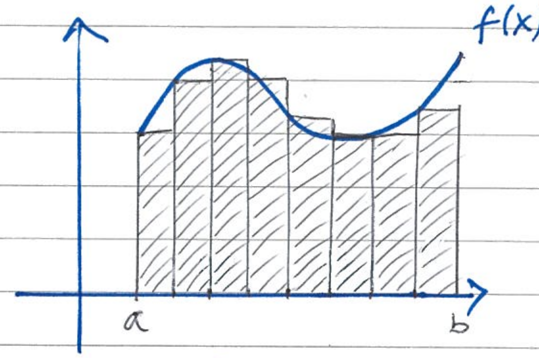
$= 1\left[\frac{1}{2}(0) + 2 + 6 + 12 + \frac{1}{2}(20)\right]$

$= \boxed{30}$

The idea is that by using more and more rectangles or trapezoids, we can make the approximation better.



n=4 rectangles



n=8 rectangles

So... taking infinitely many gives a perfect approximation!

$$\text{Exact area} = \lim_{n\to\infty} \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

i.e., $$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]}$$

However, this *limit* is <u>EXTREMELY HARD</u> to calculate, even for simple $f(x)$.

If only there was an easier way...