§8.2 - Substitution Rule (a.k.a, reverse chain rule)

- Method for doing more complicated integrals.
- Useful when we see a function and its derivative

Remember the chain rule? It tells us that

$$\left((x^2+1)^7\right)' = 7(x^2+1)^6 \cdot 2x = 14x(x^2+1)^6$$

So...

$$\int 14x(x^2+1)^6\,dx = (x^2+1)^7 + C$$

Q: How do we do this when we don't already know the answer??

A: Substitute a new variable to make things nicer!

For $\int 14x \underbrace{(x^2+1)}_{=u}{}^6 \underbrace{dx}_{=\frac{du}{2x}}$,

let $u = x^2+1$

so $du = 2x\,dx$ (take derivatives)

$\Rightarrow dx = \dfrac{du}{2x}$ (solve for dx)

$$= \int 14\cancel{x} \cdot u^6 \cdot \frac{du}{2\cancel{x}}$$

$$= \int 7u^6\,du$$

$$= u^7 + C \quad \text{(now switch back to x)}$$

$$= \boxed{(x^2+1)^7 + C}$$

NICE!

Ex:

1. $\int x^2 \sqrt{x^3+1}\, dx$

Let $u = x^3 + 1$.

$du = 3x^2\, dx. \Rightarrow dx = \frac{du}{3x^2}$

$= \int x^2 \sqrt{u}\, \frac{du}{3x^2}$

$= \frac{1}{3} \int u^{1/2}\, du$

$= \frac{1}{3}\left(\frac{2u^{3/2}}{3}\right) + C \quad = \boxed{\frac{2}{9}(x^3+1)^{3/2} + C}$

2. $\int \sin(3x)\, dx$

Let $u = 3x$

$du = 3dx \Rightarrow dx = \frac{du}{3}$

$= \int \sin(u) \cdot \frac{du}{3}$

$= \frac{1}{3} \int \sin(u)\, du$

$= \frac{-1}{3} \cos(u) + C \quad = \boxed{\frac{-1}{3} \cos(3x) + C}$

General Strategy:

- Let $u =$ _____, then $du =$ _____
- Solve for dx
- Replace u and dx terms, try to eliminate all x's.

Good Substitution Ideas:

1. u = function inside ugly power
2. u = function inside sin, cos.
3. u = function inside exponential
4. u = function under square root.
5. u = function whose derivative is also present.

Ex:

1. $\int e^x \cdot \cos(e^x)\, dx$

Let $u = e^x$

$du = e^x dx \Rightarrow dx = \frac{du}{e^x}$

$= \int \cancel{e^x} \cos(u) \frac{du}{\cancel{e^x}}$

$= \int \cos(u)\, du$

$= \sin(u) + C = \boxed{\sin(e^x) + C}$

2. $\int x^2 e^{x^3} dx$

Let $u = x^3$

$du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$

$= \int \cancel{x^2} e^u \cdot \frac{du}{3\cancel{x^2}}$

$= \int \frac{1}{3} e^u\, du$

$= \frac{1}{3} e^u + C = \boxed{\frac{1}{3} e^{x^3} + C}$

3. $\int x(x^2+1)^{2017} dx$

Let $u = x^2 + 1$

$du = 2x\, dx \Rightarrow dx = \frac{du}{2x}$

$= \int \cancel{x} (u)^{2017} \cdot \frac{du}{2\cancel{x}}$

$= \int \frac{1}{2} u^{2017} du$

$= \frac{1}{2} \cdot \frac{u^{2018}}{2018} + C = \boxed{\frac{(x^2+1)^{2018}}{4036} + C}$

4. $\int \frac{x+3}{(x^2+6x)^2} dx$.

Let $u = x^2 + 6x$

$du = (2x+6)dx \Rightarrow dx = \frac{du}{2x+6}$

$$= \int \frac{x+3}{u^2} \cdot \frac{du}{2x+6}$$

$$= \int \frac{\cancel{x+3}}{u^2} \cdot \frac{du}{2\cancel{(x+3)}}$$

$$= \frac{1}{2} \int u^{-2}\,du$$

$$= \frac{1}{2} \cdot \frac{u^{-1}}{-1} + C \quad = \boxed{\frac{-1}{2(x^2+6x)} + C}$$

Something trickier?

5. $\displaystyle\int \frac{x}{(x+7)^6}\,dx$ Let $u = x+7$ (so $x = u-7$)

$du = dx$

$$= \int \frac{x}{u^6}\,du$$

$$= \int \frac{u-7}{u^6}\,du$$

$$= \int \frac{1}{u^5} - \frac{7}{u^6}\,du$$

$$= \int u^{-5} - 7u^{-6}\,du$$

$$= \frac{u^{-4}}{-4} - \frac{7u^{-5}}{-5} + C$$

$$= \frac{-1}{4u^4} + \frac{7}{5u^5} + C \quad = \boxed{\frac{-1}{4(x+7)^4} + \frac{7}{5(x+7)^5} + C}$$

Ex (logs):

$\displaystyle\int \frac{1}{x\ln(x)}\,dx$ Let $u = \ln(x)$

$du = \frac{1}{x}\,dx$

$\Rightarrow dx = x\,du.$

$$= \int \frac{1}{\cancel{x}\cdot u} \cdot \cancel{x}\,du$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\ln(x)| + C}$$

Ex (Trig):

1. $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

$\Rightarrow dx = \frac{-du}{\sin x}$

$$= \int \frac{\cancel{\sin x}}{u} \cdot \frac{-du}{\cancel{\sin x}}$$

$$= \int \frac{-1}{u}\,du$$

$$= -\ln|u| + C = \boxed{-\ln|\cos x| + C}$$

2. $$\int \sin^5 x \cos x\,dx$$

Let $u = \sin x$

$du = \cos x\,dx$

$\Rightarrow dx = \frac{du}{\cos x}$

$$= \int u^5 \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int u^5\,du$$

$$= \frac{u^6}{6} + C = \boxed{\frac{\sin^6 x}{6} + C}$$

Ex: $$\int \frac{x^2}{\sqrt{x^3+7}}\,dx$$

Let $u = x^3+7$

$du = 3x^2\,dx$

$\Rightarrow dx = \frac{du}{3x^2}$

$$= \int \frac{\cancel{x^2}}{\sqrt{u}} \cdot \frac{du}{3\cancel{x^2}}$$

$$= \frac{1}{3}\int u^{-1/2}\,du$$

$$= \frac{1}{3}\,\frac{u^{1/2}}{1/2} + C = \frac{2}{3}u^{1/2} + C = \boxed{\frac{2}{3}(x^3+7)^{1/2} + C}$$