Chapter 8 - Integration

§8.1 - Antiderivatives

If we know $f'(x)$, how can we find $f(x)$?

Definition: $F(x)$ is an antiderivative of $f(x)$ if
$F'(x) = f(x)$.

Ex: An antiderivative of $f(x) = 4x^3$ is $F(x) = x^4$.
Why? Because $F'(x) = (x^4)' = 4x^3$!

However, this is not the only antiderivative!

For instance, $G(x) = x^4 + 2$ and $H(x) = x^4 - 1$ are antiderivatives of $f(x) = 4x^3$ too.

In general...

> If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x)$ and $G(x)$ differ by a constant.

Ex: $F(x) = x^2$, $G(x) = x^2 + 1$, and $H(x) = x^2 - \pi$ are three antiderivatives of $f(x) = 2x$.

The most general antiderivative of $f(x) = 2x$ is

$$F(x) = x^2 + C \qquad (C \in \mathbb{R})$$

This is called the indefinite integral of $f(x)$.

> Notation:
> $$\int f(x)\,dx = F(x) + C$$

integral of $f(x)$

dx = "with respect to x"

Some antiderivative of $f(x)$

Just like for derivatives, we will now develop some techniques for finding antiderivatives!

Power Rule

$$\boxed{\begin{array}{l} \text{If } n \neq -1, \text{ then } \displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \\[2em] \text{If } n = -1, \text{ then } \displaystyle\int \frac{1}{x} \, dx = \ln|x| + C \end{array}}$$

(Why absolute value?? Well... $\frac{1}{x}$ is defined for all $x \neq 0$, but $\ln x$ is defined only for $x > 0$. We can allow $x < 0$ by taking absolute values.)

Addition, Subtraction, and Constants • Functions

$$\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

$$\int k \cdot f(x) \, dx = k \cdot \int f(x) \, dx \qquad (k = \text{constant})$$

Ex:

Don't forget $+C$!!

1. $\displaystyle\int 8x^7 \, dx = \frac{\cancel{8}x^8}{\cancel{8}} + C = \boxed{x^8 + C}$

2. $\displaystyle\int x^2 + 1 \, dx = \int x^2 \, dx + \int \underset{}{\overset{= x^0}{1}} \, dx = \boxed{\frac{x^3}{3} + x + C}$

3. $\displaystyle\int -x^6 + 3x^3 - x + \pi \, dx = \boxed{-\frac{x^7}{7} + \frac{3x^4}{4} - \frac{x^2}{2} + \pi x + C}$

Sometimes you may need to rewrite the function so that our rules can be used.

Ex:

1. $\int (x^2-1)^2\,dx = \int x^4 - 2x^2 + 1\,dx$

$= \boxed{\frac{x^5}{5} - \frac{2x^3}{3} + x + C}$

2. $\int \sqrt{x} + x^{2/3}\,dx = \int x^{1/2} + x^{2/3}\,dx$

$= \frac{x^{3/2}}{3/2} + \frac{x^{5/3}}{5/3} + C$

$= \boxed{\frac{2x^{3/2}}{3} + \frac{3x^{5/3}}{5} + C}$

3. $\int \frac{x^3 + x + 2}{x^2}\,dx = \int \frac{x^3}{x^2} + \frac{x}{x^2} + \frac{2}{x^2}\,dx$

$= \int x + \frac{1}{x} + 2x^{-2}\,dx$

$= \frac{x^2}{2} + \ln|x| + \frac{2x^{-1}}{-1} + C$

$= \boxed{\frac{x^2}{2} + \ln|x| - 2x^{-1} + C}$

Exponentials:

1. $\int e^x\,dx = e^x + C$

2. $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

3. $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

4. $\int a^{kx}\,dx = \frac{a^{kx}}{k\cdot\ln(a)} + C$

Trig: 1. $\int \sin x \, dx = -\cos x + C$

2. $\int \cos x \, dx = \sin x + C$

3. $\int \sec^2 x \, dx = \tan x + C$

4. $\int \csc^2 x \, dx = -\cot x + C$

Ex:

1. $\int \cos x - \frac{1}{x} + e^{3x} \, dx = \boxed{\sin x - \ln|x| + \frac{e^{3x}}{3} + C}$

2. $\int 10^x + \sin x - 7^{3x} \, dx = \boxed{\frac{10^x}{\ln(10)} - \cos x - \frac{7^{3x}}{3 \cdot \ln(7)} + C}$

Note: We can solve for C if we have extra info.

Ex: Find $f(x)$ if $f'(x) = 3x^2 + 2$ and $f(0) = 0$.

Solution: $\int 3x^2 + 2 \, dx = x^3 + 2x + C$.

So $f(x) = x^3 + 2x + C$ for some $C \in \mathbb{R}$.

$f(0) = 0 \Rightarrow 0^3 + 2(0) + C = 0$

$\Rightarrow C = 0$

So, $\boxed{f(x) = x^3 + 2x}$

Ex: Find $f(x)$ if $f'(x) = x - \sec^2 x$ and $f(\pi) = 0$.

Solution: $\int x - \sec^2 x \, dx = \frac{x^2}{2} - \tan x + C$

Using $f(\pi) = 0$, we get $C = \frac{-\pi^2}{2}$, so $\boxed{f(x) = \frac{x^2}{2} - \tan x - \frac{\pi^2}{2}}$