§7.4 - Curve Sketching

Here we'll use our calculus knowledge to accurately sketch the graph of complicated functions.

The Process: To sketch $f(x)$...

1. Find the domain

2. Find the x-intercepts $(y=0)$, y-intercept $(x=0)$

3. Find vertical asymptotes (check limits when we $\div 0$, ln, tan, etc.)

   Find horizontal asymptotes (check $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$)

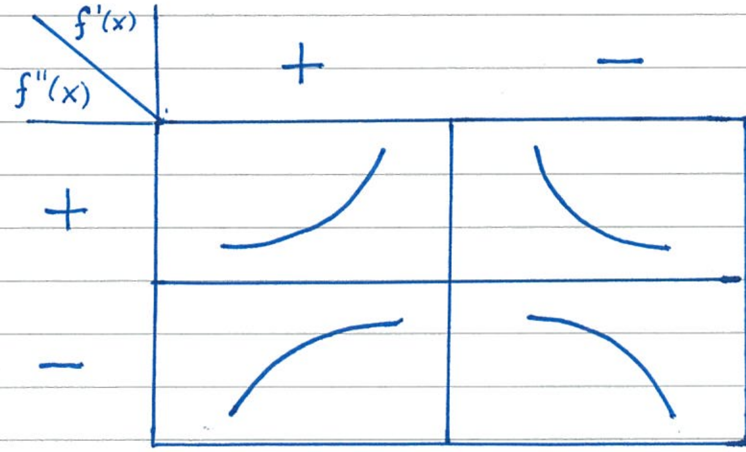
4. Find $f'(x)$ and critical points ($f'(x)=0$ or DNE)

5. Find $f''(x)$ and find where $f''(x)=0$ or DNE.

6. Make the table. Test all intervals for increase/decrease, concavity, inflection points, extrema.

7. Plot all interesting points on a graph

8. Connect them as follows:



Ex: Use calculus to sketch $f(x) = x^3 - 6x^2 + 9x$.

Solution:

1. $f(x)$ is a polynomial, so domain $= \mathbb{R}$

2. y-intercept: Set $x=0$
   Then $y = 0^3 - 6(0)^2 + 9(0) = 0$

   y-int is $(0,0)$.

   x-intecepts: Set $y=0$
   Then
$$\begin{aligned} 0 &= x^3 - 6x^2 + 9x \\ &= x(x^2-6x+9) \\ &= x(x-3)^2 \end{aligned}$$

   x-ints are $(0,0)$, $(3,0)$.

3. No vertical asymptotes.

   Horizontal asymptotes? $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$

   So no horizontal asymptotes.

4. $f'(x) = 3x^2 - 12x + 9$ (exists everywhere)

$$\begin{aligned} f'(x) = 0 &\Rightarrow 3(x^2-4x+3) = 0 \\ &\Rightarrow 3(x-3)(x-1) = 0 \\ &\Rightarrow x=1 \text{ or } x=3 \end{aligned}$$

   So $(1,4)$ and $(3,0)$ are critical points.

5. $f''(x) = 6x - 12$ (exists everywhere)

$$f''(x) = 0 \Rightarrow 6x = 12$$
$$\Rightarrow x = 2$$

So $(2,2)$ is a point of interest.

6. <u>The Table</u>:

	$x<1$	$1<x<2$	$2<x<3$	$x>3$
f''	$-$	$-$	$+$	$+$
f'	$+$	$-$	$-$	$+$
f	$\cap \nearrow$	$\cap \searrow$	$\cup \searrow$	$\cup \nearrow$
Shape				

So... $(1,4)$ is a local max
$(3,0)$ is a local min
$(2,2)$ is an inflection point.

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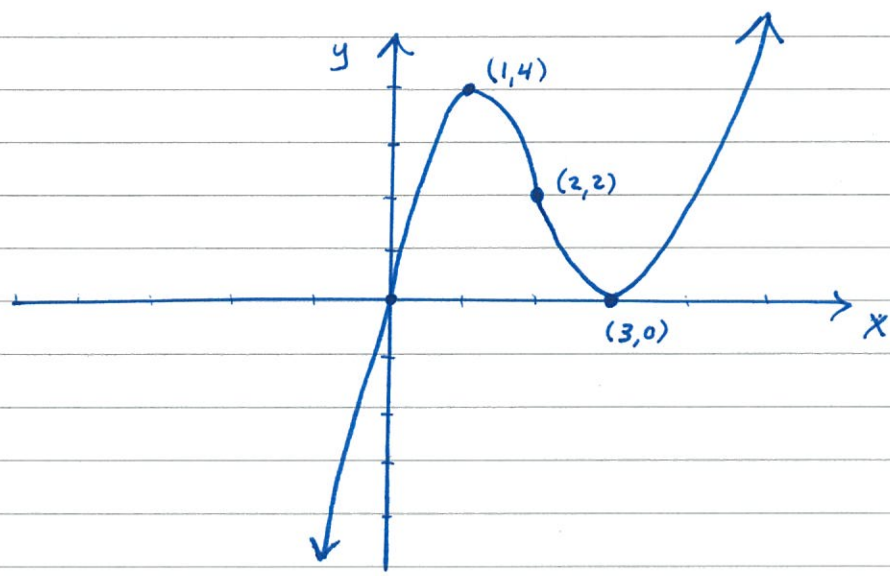

Ex: Use calculus to sketch $y = \dfrac{x^2}{x^2-4}$.

Solution:

1. Denominator $= 0$ when $x = \pm 2$, so

$$\text{Domain} = \{x \in \mathbb{R} : x \neq \pm 2\}$$

2. y-intercept: Set $x = 0$

Then $y = \dfrac{0^2}{0^2-4} = 0$

y-int is $(0,0)$

x-intercepts: Set $y = 0$

Then $0 = \dfrac{x^2}{x^2-4}$, So $x^2 = 0$.

x-int is $(0,0)$.

3. Denominator $= 0$ when $x = \pm 2$, but the numerator is not $= 0$ here!

So, we have vertical asymptotes at $x = -2$ and $x = 2$

(In general, there is a vertical asymptote whenever $f(x)$ has an infinite discontinuity (see §5.2))

Horizontal asymptotes?

$$\lim_{x\to\infty} \frac{x^2}{x^2-4} = \lim_{x\to\infty} \cancel{\frac{x^2}{x^2}}\left(\frac{1}{1-\underbrace{4/x^2}_{\to 0}}\right) = 1$$

$$\lim_{x\to-\infty} \frac{x^2}{x^2-4} = 1 \text{ as well} \Rightarrow \text{H.A. at } y = 1.$$

<u>Note</u>: A function can have 0, 1, or 2 HA's, so we have to check both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

4. $$f'(x) = \frac{(x^2-4)\cdot 2x \;-\; x^2(2x)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

$f'(x)$ DNE when $x = \pm 2$ (but neither does $f(x)$)
and $f'(x) = 0$ when $x = 0$

Critical point: $(0,0)$.

5. $$f''(x) = \frac{-8(x^2-4)^2 + 8x\left[(x^2-4)^2\right]'}{(x^2-4)^4}$$
$$= \frac{-8(x^2-4)^2 + 8x\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4}$$
$$= \frac{32x^2 - 8(x^2-4)}{(x^2-4)^3}$$
$$= \frac{8(3x^2+4)}{(x^2-4)^3}$$

So $f''(x)$ DNE at $x = \pm 2$, but $f''(x)$ is never $= 0$
(why? Because $3x^2+4$ is never $= 0$!)

6. <u>The Table</u>:

	$x<-2$	$-2<x<0$	$0<x<2$	$x>2$
	-2	0	2	
f''	+	−	−	+
f'	+	+	−	−
f	$\cup\nearrow$	$\cap\nearrow$	$\cap\searrow$	$\cup\searrow$

Shape

Our table shows that there is a local max at $x=0$.

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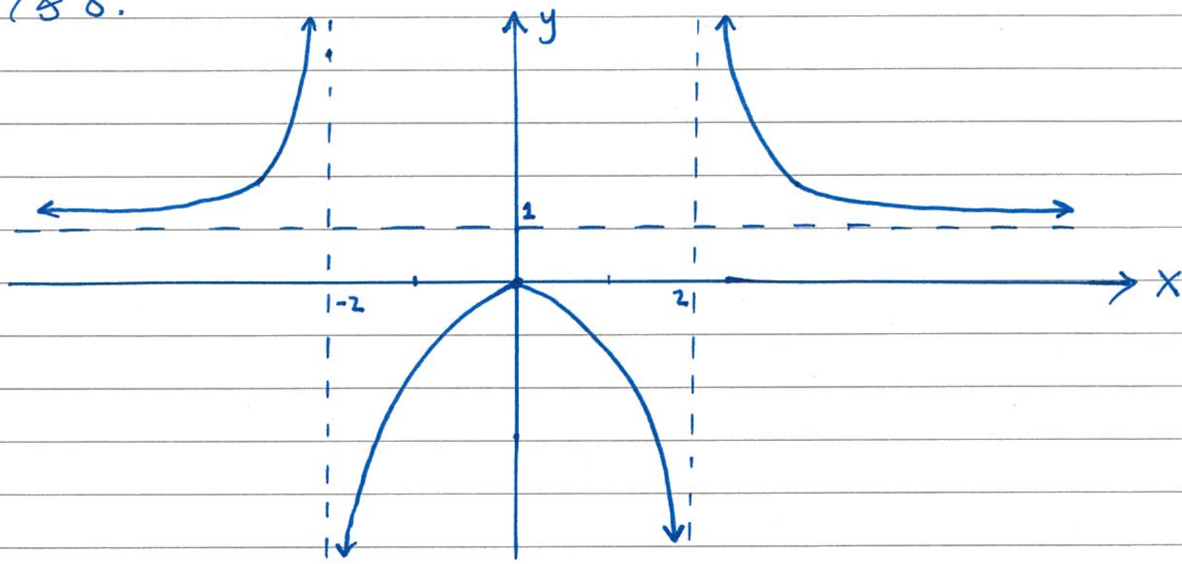


<u>Ex:</u> Use calculus to sketch $y = \dfrac{2e^x}{1+e^x}$

<u>Solution:</u>

1. Since $e^x > 0$, denominator is never $= 0$.
   So, domain $= \mathbb{R}$.

2. <u>y-intercept:</u> Set $x=0$
   Then $y = \dfrac{2e^0}{1+e^0} = \dfrac{2}{2} = 1$

   y-int is $(0, 1)$.

   <u>x-intercepts:</u> Set $y=0$
   Then $\dfrac{2e^x}{1+e^x} = 0 \Rightarrow 2e^x = 0$
   (No solution)

   So, no x-ints.

3. Since denominator $\neq 0$, no vertical asymptotes.

Horizontal?

$$\lim_{x\to\infty} \frac{2e^x}{1+e^x} = \lim_{x\to\infty} \cancel{\frac{e^x}{e^x}} \cdot \left(\frac{2}{\underbrace{1/e^x}_{\to 0} + 1}\right) = \frac{2}{0+1} = 2$$

$$\lim_{x\to -\infty} \frac{2e^x}{1+e^x} = \frac{2(0)}{1+0} = 0$$

So, we have HA's at $y=2$ and $y=0$.

4. $$f'(x) = \frac{(1+e^x)\cdot 2e^x - 2e^x \cdot e^x}{(1+e^x)^2}$$
$$= \frac{2e^x}{(1+e^x)^2}$$

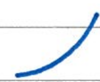
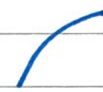
$f'(x)$ exists everywhere and never $= 0$.
$\Rightarrow$ No critical points!

5. $$f''(x) = \frac{(1+e^x)^2 \cdot 2e^x - 2e^x\left[(1+e^x)^2\right]'}{(1+e^x)^4}$$
$$= \frac{(1+e^x)^2 \cdot 2e^x - 2e^x \cdot 2(1+e^x)\cdot e^x}{(1+e^x)^4}$$
$$= \frac{(1+e^x)\cdot 2e^x - 4e^{2x}}{(1+e^x)^3}$$
$$= \frac{2e^x\cdot(1-e^x)}{(1+e^x)^3}$$

$f''(x)$ exists everywhere
$f''(x) = 0$ when $1-e^x = 0$ (i.e, when $x=0$)

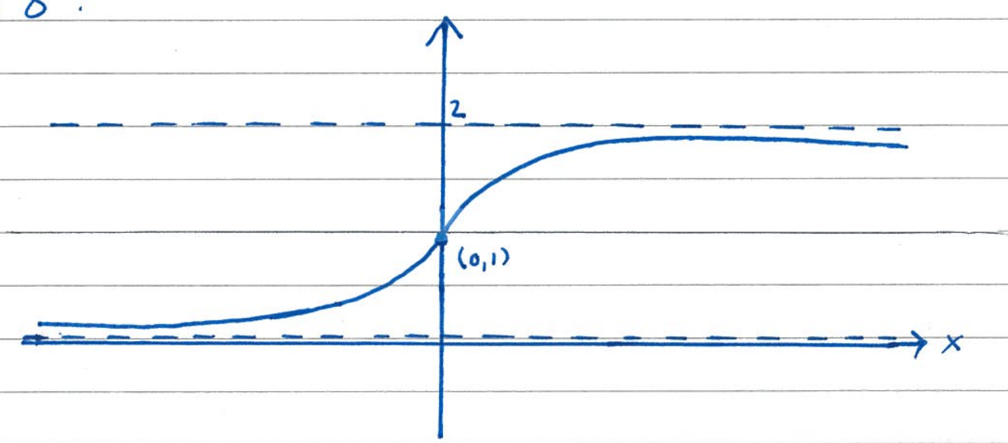
So $(0, 1)$ is a point of interest.

6. The Table:

		0	
f''	$+$		$-$
f'	$+$		$+$
f	$\cup\nearrow$		$\cap\nearrow$
Shape			

(Inflection point at $(0,1)$)

7 & 8:



~ END OF DIFFERENTIAL CALCULUS