§7.3 - Higher Derivatives; Concavity

Why stop at just 1 derivative?

The second derivative of $f(x)$ is $f''(x)$, given by

$$\boxed{f''(x) = (f'(x))'}$$

The third derivative of $f(x)$ is $f'''(x)$, given by

$$\boxed{f'''(x) = (f''(x))'}$$

After 3, we write $f^{(4)}(x)$, $f^{(5)}(x)$, $f^{(6)}(x)$, etc. for higher derivatives.

Ex: If $f(x) = x^3 - 7x^2 + 2x + 3$, then

$$f'(x) = 3x^2 - 14x + 2$$
$$f''(x) = 6x - 14$$
$$f'''(x) = 6$$
$$f^{(4)}(x) = 0 \qquad \text{etc.}$$

Ex: If $g(x) = \sin x + x \ln x$, then

$$g'(x) = \cos x + \left(\ln x + x \cdot (1/x)\right) \qquad \text{(product rule)}$$
$$= \cos x + \ln x + 1$$

$$g''(x) = -\sin x + \frac{1}{x}$$

$$g'''(x) = -\cos x - \frac{1}{x^2}$$

$$g^{(4)}(x) = \sin x + \frac{2}{x^3} \qquad \text{etc.}$$

An Application: Distance, Velocity, Acceleration

Suppose $f(t)$ = distance travelled at time t.

Then $f'(x)$ = velocity at time t
$f''(x)$ = acceleration at time t

The higher derivatives have names too!

$f'''(t)$ = jerk
$f^{(4)}(t)$ = jounce (or snap)
$f^{(5)}(t)$ = crackle
$f^{(6)}(t)$ = pop
$f^{(7)}(t)$ = lock
$f^{(8)}(t)$ = drop. (You don't need to know these.)

Ex: A runner's distance at t seconds is given by $f(t) = t^4 + 6t^2$. What is her acceleration at $t = 2$ seconds?

Solution: velocity $= f'(t) = 4t^3 - 12t$

acceleration $= f''(t) = 12t^2 - 12$

At $t=2$, acceleration is $12 \cdot 4 - 12 = \boxed{36 \text{ m/s}^2}$

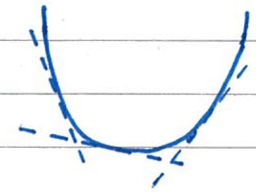
(fast runner!)

Concavity

The second derivative is the rate of change of the first derivative.

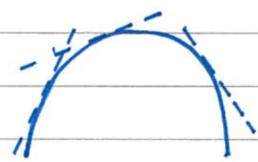
So, if $f''(x) > 0$, then $f'(x)$ is increasing, so the slopes of the tangent lines are increasing.

$f(x)$ would look something like



Concave up

On the other hand, if $f''(x) < 0$ then $f'(x)$ is decreasing, so $f(x)$ looks something like

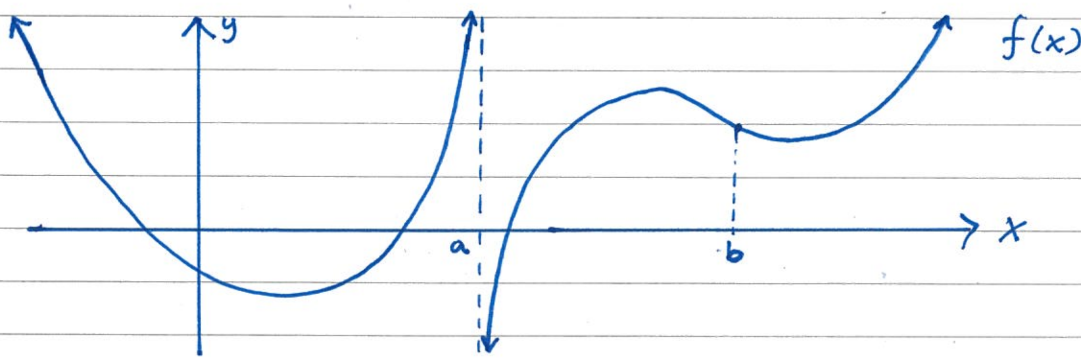


Concave down

So $f(x)$ is <u>concave up</u> on an interval I if $f''(x) > 0$.
<u>concave down</u> on an interval I if $f''(x) < 0$.

A point (in the domain of $f(x)$) where concavity changes is called an <u>inflection point</u>.

<u>Ex:</u>



$f(x)$ is concave up on $(-\infty, a) \cup (b, \infty)$,
concave down on (a, b).

$x = b$ is an inflection point ($x = a$ is NOT (not in domain)).

<u>Ex:</u> Find all inflection points and intervals of concavity for $f(x) = 3x^5 + 5x^4 - 20x^3 + 4$.

<u>Solution:</u> $f'(x) = 15x^4 + 20x^3 - 60x^2$

$$f''(x) = 60x^3 + 60x^2 - 120x$$
$$= 60x(x^2 + x - 2) = 60x(x+2)(x-1)$$

So $f''(x) = 0$ when $x = 0, -2, 1$.
We'll test concavity around these points.

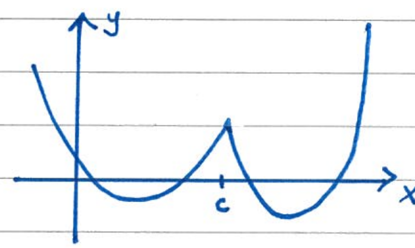
<u>Test</u>:

	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f''(x)$	$-$		$+$		$-$		$+$
$f(x)$	$\cap$		$\cup$		$\cap$		$\cup$

So $f(x)$ is concave up on $(-2, 0) \cup (1, \infty)$, concave down on $(-\infty, -2) \cup (0, 1)$, and $x = -2$, $x = 0$, $x = 1$ are inflection pts.

<u>Note</u>: If $x = c$ is an inflection point, then either $f''(c) = 0$ or $f''(c)$ DNE.

However, just because $f''(c) = 0$ or $f''(c)$ DNE does NOT mean $x = c$ is an inflection point!!



$f''(c)$ DNE but $x = c$ is not an inflection point.

We need to make sure concavity changes!

The second derivative gives us an alternate test for max's and min's.

<u>2nd Derivative Test</u>: If $f''(x)$ exists at $x = c$ and $f'(c) = 0$ (so $x = c$ is a critical point) then

(1) $f''(c) > 0 \Rightarrow x = c$ is a local min.

(2) $f''(c) < 0 \Rightarrow x = c$ is a local max

(3) $f''(c) = 0 \Rightarrow$ No information.

Note: The First Derivative Test work just as well. Use whatever test you'd like.

Ex: If $f(x) = x^2$, then $f'(x) = 2x$ and $f''(x) = 2$.

Since $f'(0) = 0$ and $f''(0) = 2 > 0$, $x = 0$ is a local min by the 2^{nd} derivative test.

Exercise: Show that $g(x) = x^4 - 2x^2$ has a local max at $x = 0$ and local mins at $x = \pm 1$.

Ex: Find where $f(x)$ is concave up/down. List any inflection points.

(1) $f(x) = x^4 - 6x^2$.

Solution: $f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$ ($= 0$ at $x = \pm 1$)

Test

$f''(x)$: $+$ | -1 | $-$ | 1 | $+$

$f(x)$: $\cup$ $\cap$ $\cup$

Concave up: $(-\infty, -1) \cup (1, \infty)$
Concave down: $(-1, 1)$
inflection pts: $x = \pm 1$

(2) $f(x) = \dfrac{1}{x}$

Solution: $f'(x) = \dfrac{-1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$ (never $= 0$, but DNE at $x = 0$)

Test:

$f''(x)$: $-$ | 0 | $+$

$f(x)$: $\cap$ $\cup$

Concave up: $(0, \infty)$
concave down: $(-\infty, 0)$
inflection pts: None

($x = 0$ not in domain)