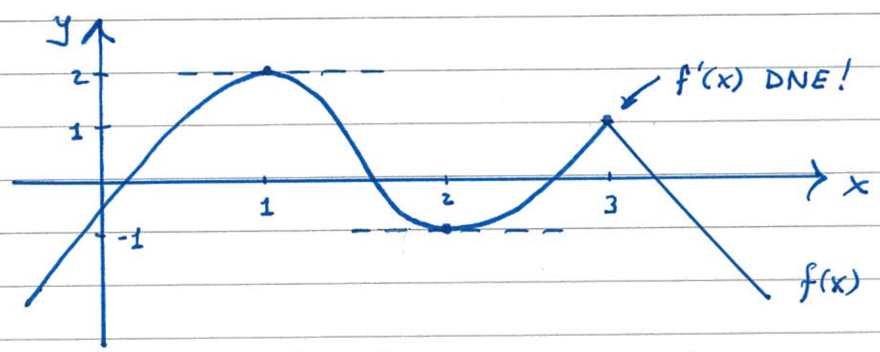
§7.2 - Relative Extrema (Max / Min)

Let c be in the domain of $f(x)$. Then $f(x)$ is said to have

- a **local max** at c with value $f(c)$ if $f(x) \le f(c)$ for all x near c;
- a **local min** at c with value $f(c)$ if $f(x) \ge f(c)$ for all x near c.

Ex.

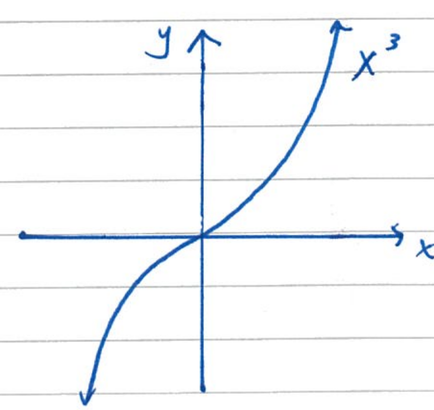


$f(x)$ has a
- local max at $x = 1$ with value 2.
- local min at $x = 2$ with value -1
- local max at $x = 3$ with value 1.

Note: A max / min seems to occur at critical points...

But not every critical point is a max or a min!

Ex: $f(x) = x^3$ has no local max or min, even though it has a critical point at $x = 0$!



We can find max's and min's using the

First Derivative Test: Let c be a critical point of $f(x)$.

(1) If $f'(x)$ changes from $+$ to $-$ at $x=c$, then $f(x)$ has a <u>local max</u> at $x=c$.

(2) If $f'(x)$ changes from $-$ to $+$ at $x=c$, then $f(x)$ has a <u>local min</u> at $x=c$.

(3) If the sign of $f'(x)$ doesn't change at $x=c$, then $x=c$ is <u>neither</u> a max nor a min.

Just remember...

$f'(x)$ $+$ C.P. $-$
$f(x)$ $\nearrow$ $\searrow$
(local max)

$f'(x)$ $-$ C.P. $+$
$f(x)$ $\searrow$ $\nearrow$
(local min)

$f'(x)$ $+$ C.P $+$
$f(x)$ $\nearrow$ $\nearrow$
(neither)

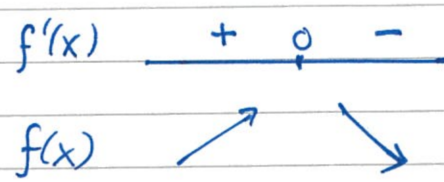
$f'(x)$ $-$ C.P. $-$
$f(x)$ $\searrow$ $\searrow$
(neither)

Ex: Find all local maxima and local minima.

(1) $f(x) = e^{-x^2}$

Solution: $f'(x) = -2x\underbrace{e^{-x^2}}_{>0 \text{ always}}$ $\Rightarrow$ $f'(x)$ exists everywhere, $f'(x)=0$ at $x=0$.

Only critical point: $x = 0$.

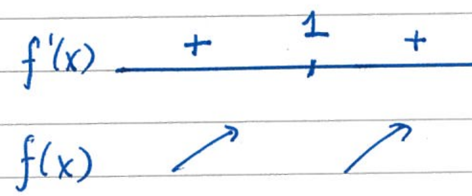


Testing points on either side, we see $x = 0$ is a local max with value $f(0) = 1$.

(2) $f(x) = (x-1)^5$

Solution: $f'(x) = 5(x-1)^4 \Rightarrow f'(x)$ exists everywhere, $f'(x) = 0$ at $x = 1$.

Only critical point: $x = 1$.



Testing points on either side, we see $x = 1$ is neither a local max nor a local min.

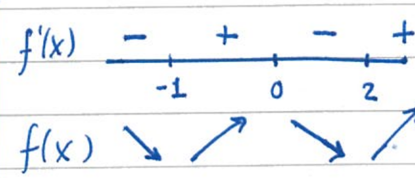
Thus, no local extrema.

(3) $f(x) = 3x^4 - 4x^3 - 12x^2$.

Solution:
$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x \\ &= 12x(x^2 - x - 2) \\ &= 12x(x-2)(x+1). \end{aligned}$$

So, $f'(x)$ exists everywhere and $f'(x) = 0$ at $x = 0$, $x = -1$, and $x = 2$.

$x = -1$ is a local min w/ value $f(-1) = -5$
$x = 0$ is a local max w/ value $f(0) = 0$
$x = 2$ is a local min w/ value $f(2) = -32$

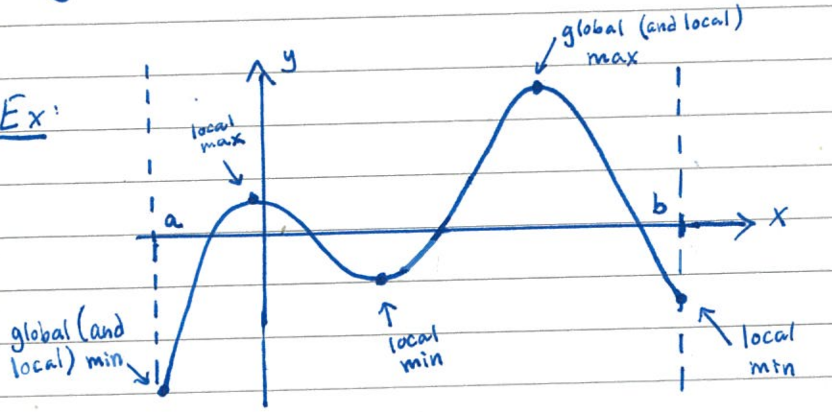


Global Extrema

A point c in the domain of $f(x)$ is called a

- <u>global max</u> if $f(c) \geq f(x)$ for all x in the domain;
- <u>global min</u> if $f(c) \leq f(x)$ for all x in the domain.

<u>Ex:</u>



★ We'll call an endpoint $x=a$ or $x=b$ a local max / min as well. Not all sources will do this... ★

We will always look for global extrema on a closed interval $[a,b]$. To find them...

(1) find all critical points inside $[a,b]$;
(2) compute $f(a)$, $f(b)$, and $f(\text{critical points})$;
(3) biggest = global max, smallest = global min.

<u>Ex:</u> Find the global max / min for $f(x) = x^3 - 3x - 2$ on $[-3,3]$.

<u>Solution:</u> Follow the 3 steps above.

$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$, so $f'(x)$ exists everywhere and $f'(x) = 0$ at $x = -1$ and $x = 1$.

$f(-3) = -20$ ← smallest
$f(-1) = 0$
$f(1) = -4$
$f(3) = 16$ ← biggest.

$$\boxed{\text{So } f(x) \text{ has a global min at } x=-3 \text{ with value } -20, \text{ and a global max at } x=3 \text{ with value } 16.}$$

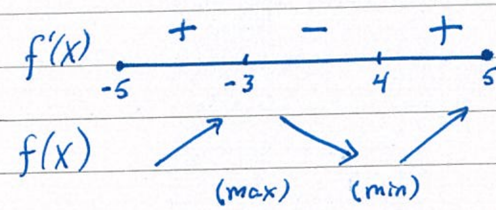
Ex: Find all local and global extrema for $f(x)$.

(1) $f(x) = 2x^3 - 3x^2 - 72x + 5$ on $[-5, 5]$.

Solution: $f'(x) = 6x^2 - 6x - 72$
$$= 6(x^2 - x - 12)$$
$$= 6(x-4)(x+3)$$

So $f'(x)$ exists everywhere and $f'(x) = 0$ when $x = -3$ or $x = 4$. (these are the critical points)

Local:



$x = -5$ is a local min
$x = -3$ is a local max
$x = 4$ is a local min
$x = 5$ is a local max.

Global:
$f(-5) = 40$
$f(-3) = 140$ ← biggest
$f(4) = -203$ ← smallest.
$f(5) = -180$

So $x = -3$ is global max with value 140, $x = 4$ is global min with value -203.

(2) $f(x) = 6x^{2/3} - 4x + 2$ on $[-1, 1/2]$

Solution: $f'(x) = 6(2/3)x^{-1/3} - 4$
$$= 4x^{-1/3} - 4$$
$$= \frac{4}{x^{1/3}} - 4$$

Note that $f'(x)$ DNE at $x = 0$. This is a critical point. Find any others by solving $f'(x) = 0$.

$$f'(x) = 0 \Rightarrow \frac{4}{x^{1/3}} = 4$$

$$\Rightarrow 4 = 4x^{1/3}$$

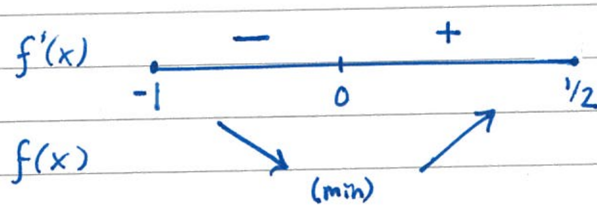
$$\Rightarrow 1 = x^{1/3}$$

$$\Rightarrow x = 1$$

(This point doesn't belong to $[-1, \frac{1}{2}]$, so we ignore it!)

<u>Local</u>:

$x = 0$ is a local min.
$x = -1$ is a local max.
$x = \frac{1}{2}$ is a local max.



<u>Global</u>: $f(-1) = 12$ ← biggest

$f(0) = 2$ ← smallest.

$f(\frac{1}{2}) \approx 3.78$ (calculator)

So $x = 0$ is a global min with value 2,
$x = -1$ is a global max with value 12.