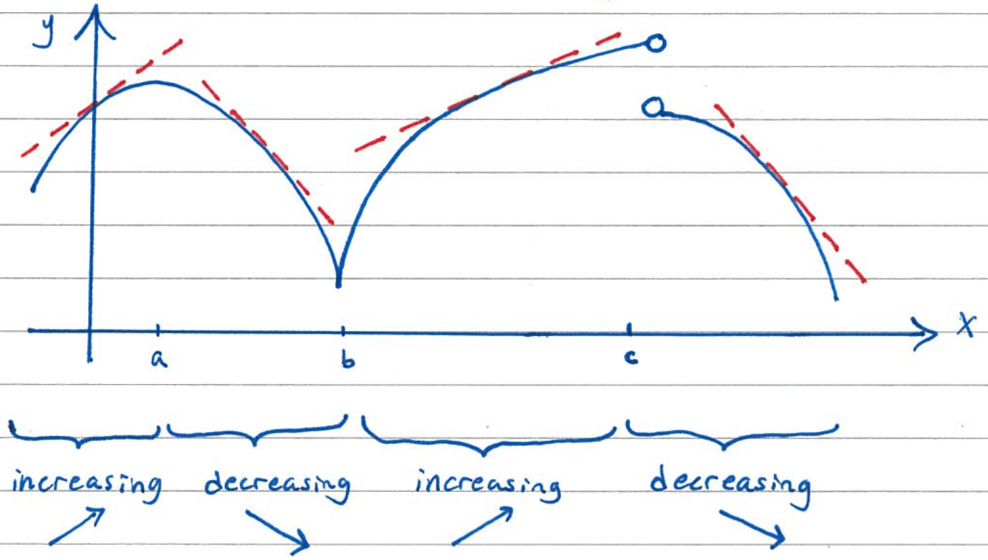
Chapter 7 - Applications of Differentiation.

Surprise! Derivatives are actually good for something. For many things, in fact. By studying $f'(x)$, we can

- better understand (and accurately graph) $f(x)$.
- Solve max/min problems, and
- solve real-world problems involving rates of change.

§7.1 - Increasing/Decreasing Functions



Formally, $f(x)$ is increasing on an interval I if for every $x_1 < x_2$ in I, $f(x_1) < f(x_2)$

Similarly, $f(x)$ is decreasing on an interval I if for every $x_1 > x_2$ in I, $f(x_1) > f(x_2)$.

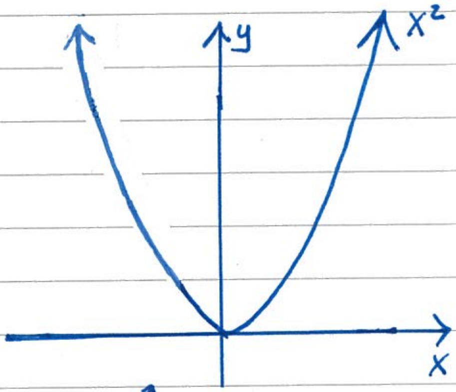
Observations:

- Tangent lines have positive slope when $f(x)$ increasing, negative slope when $f(x)$ decreasing.

- At $x=a$, $f'(x)=0$. At $x=b$, $f'(x)$ DNE.
  At $x=c$, $f(x)$ is undefined.

Suppose $f(x)$ is a function and $f'(x)$ exists on an interval I.

- If $f'(x) > 0$ for all $x \in I$, then $f(x)$ is increasing on I.
- If $f'(x) < 0$ for all $x \in I$, then $f(x)$ is decreasing on I.



Ex: If $f(x) = x^2$, then
$f'(x) = 2x$.

Notice $f'(x) > 0$ on $(0, \infty)$
and $f'(x) < 0$ on $(-\infty, 0)$.

$\Rightarrow f(x) = x^2$ increasing on $(0,\infty)$
and decreasing on $(-\infty, 0)$.

↑
(seems legit.)

Definition: A critical number (or critical point) of $f(x)$ is a point c in the domain such that either

- $f'(c) = 0$, or
- $f'(c)$ DNE.

The important thing is that a function $f(x)$ can go from increasing to decreasing (or vice versa) only at a critical point, or at a point where $f(x)$ is undefined.

This gives us a slick test for finding where a function is increasing / decreasing!

4 Step Test for Increasing / Decreasing

1. Find $f'(x)$

2. Find all critical points.

3. Plot critical points on a line, as well as any points where $f(x)$ is undefined.

4. Check if $f'(x)$ is >0 or <0 between these points.
   - $f'(x) > 0 \Rightarrow f(x)$ increasing
   - $f'(x) < 0 \Rightarrow f(x)$ decreasing

Ex: Where is $f(x) = x^3 + 9x^2 - 21x + 4$ increasing/decreasing?

Solution:

1. $f'(x) = 3x^2 + 18x - 21$

2. Note: $f'(x)$ exists everywhere.

$$f'(x) = 0 \Rightarrow 3x^2 + 18x - 21 = 0$$
$$\Rightarrow 3(x^2 + 6x - 7) = 0$$
$$\Rightarrow 3(x+7)(x-1) = 0, \quad \text{so } x = -7 \text{ or } x = 1.$$

The critical points are $x = -7$ and $x = 1$.

3.

check $f'(x)$ inside each interval.

At $x = -8$, $f'(x) > 0$
At $x = 0$, $f'(x) < 0$
At $x = 2$, $f'(x) > 0$

4. $\boxed{f(x) \text{ is increasing on } (-\infty, -7) \cup (1, \infty) \quad \text{decreasing on } (-7, 1).}$

Ex: Find where $f(x)$ is increasing/decreasing and list any critical points.

(1) $f(x) = e^{x^2}$.

Solution: Follow the 4-step test.

$$f'(x) = e^{x^2}\cdot(x^2)' = 2x\cdot e^{x^2} \quad \text{(this exists everywhere!)}$$

$$f'(x) = 0 \Rightarrow 2x\cdot \underbrace{e^{x^2}}_{>0 \text{ always}} = 0 \Rightarrow 2x = 0 \Rightarrow \boxed{x=0}$$

(our only critical point)

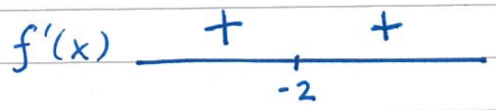
So $\boxed{f(x) \text{ is increasing on } (0,\infty) \text{ and decreasing on } (-\infty,0).}$

At $x=-1$, $f'(x) < 0$
At $x=+1$, $f'(x) > 0$

(2) $f(x) = \dfrac{x-1}{x+2}$

Solution: $f'(x) = \dfrac{(x-1)'(x+2)-(x-1)(x+2)'}{(x+2)^2} = \dfrac{3}{(x+2)^2}$

$f'(x)$ is never 0, but $f'(x)$ DNE at $x=-2$.
(however, this is not a critical point: it doesn't belong to the domain!)



So $\boxed{f(x) \text{ is increasing on } (-\infty,-2) \cup (-2,\infty)}$

At $x=-3$, $f'(x) > 0$
At $x=0$, $f'(x) > 0$

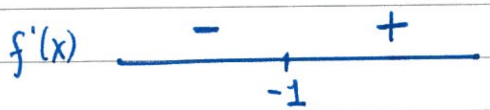
Note: There are no critical points

(3) $f(x) = (x+1)^{2/3}$

Solution: $f'(x) = \frac{2}{3}(x+1)^{-1/3} = \frac{2}{3(x+1)^{1/3}}$.

$f'(x)$ is never 0, but $f'(x)$ DNE at $x=-1$. (This time this is a critical point, as $x=-1$ belongs to the domain of $f(x)$!)

So, our critical point is $\boxed{x=-1}$



So, f(x) is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$.

At $x=-2$, $f'(x)<0$
At $x=0$, $f'(x)>0$