§6.6 - Derivatives of Trigonometric Functions

Goal: Determine the derivatives of the trig functions:
$\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$, $\cot x$.

We'll need the following useful identities:

- $\sin^2 x + \cos^2 x = 1$

- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

- $\tan x = \dfrac{\sin x}{\cos x}$
- $\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$.

Another very useful fact: $\boxed{\lim_{x \to 0} \dfrac{\sin x}{x} = 1.}$

Using this, we can show that $\boxed{\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0}$

Indeed,
$$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$$

$$= \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \qquad (= -\sin^2 x)$$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} = 1 \cdot \frac{0}{2} = \boxed{0}$$

Putting these ingredients together, we get

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \sin x \underbrace{\left(\frac{\cos h - 1}{h}\right)}_{=0} + \cos x \underbrace{\left(\frac{\sin h}{h}\right)}_{=1}$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \cos x$$

So... $\boxed{(\sin x)' = \cos x.}$

<u>What about $(\cos x)'$?</u>

Well, $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$.

Take derivatives (chain rule!):

$$2\cos x\,(\cos x)' = -2\sin x\,(\sin x)'$$
$$= -2\sin x \cos x$$

$$\Rightarrow (\cos x)' = \frac{-2\sin x \cos x}{2\cos x} = -\sin x$$

So... $\boxed{(\cos x)' = -\sin x}$

<u>Ex:</u> Find the derivative:

(1) $f(x) = \sin(6x)$

<u>Solution:</u> $f'(x) = \cos(6x) \cdot (6x)' = \boxed{6\cos(6x)}$

(2) $f(x) = \cos(\sin x)$

Solution: $f'(x) = -\sin(\sin x) \cdot (\sin x)' = \boxed{-\sin(\sin x) \cdot \cos x}$

chain rule!

(3) $f(x) = x^2 \cos x$

Solution: $f'(x) = (x^2)' \cdot \cos x + x^2 \cdot (\cos x)'$

$$= \boxed{2x \cdot \cos x - x^2 \cdot \sin x.}$$

(4) $f(x) = e^{\sin x}$

Solution: $f'(x) = e^{\sin x} \cdot (\sin x)' = \boxed{e^{\sin x} \cdot \cos x.}$

(5) $f(x) = x^{\sin x}$

Solution: This has the form $g(x)^{h(x)}$ ... log differentiation!

[Apply ln]: $\ln f(x) = \sin x \cdot \ln x$

[Take derivatives]: $\dfrac{f'(x)}{f(x)} = (\sin x)' \cdot \ln x + \sin x \cdot (\ln x)'$

$$= \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

[Solve for $f'(x)$]: $f'(x) = f(x)\left(\cos x \cdot \ln x + \dfrac{\sin x}{x}\right)$

$$= \boxed{x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)}$$

What about $\tan x$, $\sec x$, $\csc x$, and $\cot x$?

Well... $\tan x = \dfrac{\sin x}{\cos x}$, so use quotient rule.

$$(\tan x)' = \frac{(\sin x)'\cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

So... $\boxed{(\tan x)' = \sec^2 x}$

Here's another one:

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{(1)' \cdot \cos x - 1 \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{0 \cdot \cos x - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \boxed{\sec x \cdot \tan x.}$$

So... $\boxed{(\sec x)' = \sec x \cdot \tan x}$

In the tutorial, you will repeat these arguments to show that

(1) $\boxed{(\cot x)' = -\csc^2 x}$

(2) $\boxed{(\csc x)' = -\csc x \cdot \cot x.}$

To summarize:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cdot \cot x$

Ex: Find the derivative:

(1) $f(x) = \cot(9x^2)$

Solution: $f'(x) = -\csc^2(9x^2) \cdot (9x^2)' = \boxed{-\csc^2(9x^2) \cdot 18x}$

chain rule!

(2) $f(x) = e^x \cdot \sec(x^2)$.

Solution: $f'(x) = (e^x)' \sec(x^2) + e^x \left(\sec(x^2)\right)'$ (product rule)

$= e^x \cdot \sec(x^2) + e^x \cdot \sec(x^2) \cdot \tan(x^2) \cdot (x^2)'$

$= \boxed{e^x \cdot \sec(x^2) + e^x \sec(x^2) \cdot \tan(x^2) \cdot 2x}$

(3) $f(x) = \cos(\ln(\tan x))$

Solution: $f'(x) = -\sin(\ln(\tan x)) \cdot (\ln(\tan x))'$

$= -\sin(\ln(\tan x)) \cdot \dfrac{1}{\tan x} \cdot (\tan x)'$

$= \boxed{-\sin(\ln(\tan x)) \cdot \dfrac{\sec^2 x}{\tan x}}$