§ 6.5 - Derivatives of Logarithmic Functions

We know that $(e^x)' = e^x$ and $(e^{f(x)})' = e^{f(x)} \cdot f'(x)$.

New question: What's the derivative of $f(x) = \ln x$?

Well... $f(x) = \ln x \Rightarrow e^{f(x)} = e^{\ln x} = x$

$$\Rightarrow (e^{f(x)})' = (x)'$$

$$\Rightarrow e^{f(x)} \cdot f'(x) = 1$$

Thus, $f'(x) = \frac{1}{e^{f(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x} \quad (x > 0)$.

$$\boxed{\text{The derivative of } \ln x \text{ is } \tfrac{1}{x} \ (x>0).}$$

(This is what we observed graphically in §5.5 !)

Ex: Find the derivative.

(1) $f(x) = x \ln x$.

Solution: $f'(x) = (x)' \ln x + x(\ln x)'$ (product rule)

$$= \ln x + x\left(\frac{1}{x}\right)$$

$$= \boxed{\ln x + 1}$$

(2) $f(x) = \dfrac{x^2+1}{\ln x}$

Solution: $f'(x) = \dfrac{(x^2+1)' \ln x - (x^2+1)(\ln x)'}{(\ln x)^2}$

$$= \boxed{\frac{2x \ln x - \frac{x^2+1}{x}}{(\ln x)^2}}$$

(3) $f(x) = \ln(x^2)$

Solution: 2 Methods...

Chain rule: $f'(x) = \frac{1}{x^2}(x^2)' = \frac{2x}{x^2} = \boxed{\frac{2}{x}}$

Cheap trick: $f(x) = \ln(x^2) = 2\ln x$, so $f'(x) = \boxed{\frac{2}{x}}$

In general, the chain rule shows that

$$\boxed{\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)}} \qquad (*)$$

What about logs with different bases?

Well... since $\log_a x = \frac{\ln x}{\ln a}$, we have $\boxed{(\log_a x)' = \frac{1}{x \ln a}}$

For general bases $a > 0$, rule $(*)$ above becomes

$$\boxed{\left(\log_a(f(x))\right)' = \frac{f'(x)}{f(x) \cdot \ln a}}$$

Ex: Find the derivative.

(1) $f(x) = \log_2(x^2 + x)$

Solution: Method 1 (chain rule):

$$f'(x) = \frac{(x^2+x)'}{(x^2+x)\ln 2} = \boxed{\frac{2x+1}{(x^2+x)\ln 2}}$$

Method 2 (Log properties):

$$f(x) = \log_2(x^2+x) = \log_2(x(x+1)) = \log_2 x + \log_2(x+1)$$

So $f'(x) = \boxed{\dfrac{1}{x \ln 2} + \dfrac{1}{(x+1)\ln 2}}$

(2) $f(x) = e^x \cdot \log_5(x^3-1).$

Solution: $f'(x) = (e^x)' \log_5(x^3-1) + e^x\left(\log_5(x^3-1)\right)'$

$$= e^x \log_5(x^3-1) + \frac{e^x \cdot (x^3-1)'}{(x^3-1)\cdot \ln 5}$$

$$= \boxed{e^x \log_5(x^3-1) + \frac{3x^2 e^x}{(x^3-1)\ln 5}}$$

(3) $f(x) = (x \log_{10}(x))^3.$

Solution: $f'(x) \underset{\text{chain rule.}}{=} 3(x\log_{10}(x))^2 \cdot \underbrace{(x\log_{10}(x))'}_{\text{product rule}}$

$$= 3(x\log_{10}(x))^2\left(\log_{10}(x) + x \cdot \frac{1}{x \cdot \ln(10)}\right)$$

$$= \boxed{3(x\log_{10}(x))^2\left(\log_{10}(x) + \frac{1}{\ln(10)}\right)}$$

An Application: Logarithmic Differentiation.

We know how to find the derivative of a function to the power of a number:

$$(f(x)^n)' = n(f(x))^{n-1} \cdot f'(x)$$

We also know how to find the derivative of a number to the power of a function:

$$\left(a^{f(x)}\right)' = a^{f(x)} \cdot \ln a \cdot f'(x)$$

But what about a function to the power of a function?

$$f(x)^{g(x)} = ??$$

We use a process called <u>logarithmic differentiation.</u>

3 steps in log diff.	Ex: If $f(x) = x^x$, find $f'(x)$.
(1) Take ln on both sides.	$\ln(f(x)) = \ln(x^x) = x \cdot \ln(x).$
(2) Differentiate both sides.	$\frac{f'(x)}{f(x)} = \ln(x) + x \cdot \frac{1}{x}$ (prod. rule) $= \ln(x) + 1.$
(3) Solve for $f'(x)$ You're Done!!	$f'(x) = f(x)\left(\ln(x) + 1\right)$ $= \boxed{x^x\left(\ln(x) + 1\right)}$

<u>Ex:</u> Find $f'(x)$ when $f(x) = x^{(e^x)}$.

<u>Solution:</u> Follow the 3 steps above.

(1) $\ln(f(x)) = \ln\left(x^{(e^x)}\right) = e^x \ln(x)$

(2) $\dfrac{f'(x)}{f(x)} = (e^x)' \ln x + e^x (\ln x)'$

$= e^x \ln x + \dfrac{e^x}{x}$

(3) $f'(x) = f(x)\left(e^x \ln x + \dfrac{e^x}{x}\right) = \boxed{x^{(e^x)}\left(e^x \ln x + \dfrac{e^x}{x}\right)}$

Ex: Find $f'(x)$ when $f(x) = (\ln x)^{5x^2}$

This is NOT $5x^2 \ln(x)$!

Solution: Use log. differentiation.

(1) $\ln f(x) = 5x^2 \ln(\ln x)$

(2) $\dfrac{f'(x)}{f(x)} = (5x^2)' \cdot \ln(\ln x) + 5x^2 (\ln(\ln x))'$

$= 10x \ln(\ln x) + 5x^2 \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$

(3) $f'(x) = f(x)\left(10x \ln(\ln x) + \dfrac{5x}{\ln x}\right)$

$= \boxed{(\ln x)^{5x^2}\left(10x \ln(\ln x) + \dfrac{5x}{\ln x}\right)}$

Ex: Find $f'(x)$ when $f(x) = 3^{(7^x)}$.

Solution: This is just $a^{g(x)}$!

$f'(x) = 3^{(7^x)} \cdot \ln 3 \cdot (7^x)' = \boxed{3^{(7^x)} \cdot \ln 3 \cdot (7^x \cdot \ln 7)}$