# §6.4 - Derivatives of Exponential Functions.

Remember "e"? That constant from Chapter 4?
We defined e as

"the unique constant such that $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$".

This exactly says that $f'(0) = 1$. That is,

$$1 = f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}$$

So $$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

Fine... but what's so good about this??

Answer: It allows us to calculate $f'(x)$!

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\ &= e^x \underbrace{\left[\lim_{h \to 0} \frac{e^h - 1}{h}\right]}_{=1} = \boxed{e^x} \end{aligned}$$

Yup! The derivative of $e^x$ is... $e^x$!!

Ex: Find the derivative:

(1) $f(x) = 4e^x$

Solution: $f'(x) = \boxed{4e^x}$

(2) $f(x) = x \cdot e^x$

product rule!

Solution: $f'(x) = (x)'e^x + x(e^x)' = \boxed{e^x + xe^x}$

(3) $f(x) = e^{\sqrt{x}}$

chain rule!

Solution: $f'(x) = e^{\sqrt{x}} \cdot (\sqrt{x})' = \boxed{e^{\sqrt{x}} \cdot \left(\frac{1}{2}x^{-1/2}\right)}$

In fact, the chain rule implies that

$$\boxed{\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)}$$

So, if $f(x) = e^{10x^3+4}$, then

$$f'(x) = e^{10x^3+4} \cdot (10x^3+4)' = \boxed{e^{10x^3+4} \cdot 30x^2}$$

What about $a^x$? Well, notice that $a = e^{\ln(a)}$, so

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}.$$

Can we do it now? Use the chain rule!

$$(a^x)' = \left(e^{x\ln a}\right)' = \underbrace{e^{x\ln a}}_{=a^x} \cdot \underbrace{(x\ln a)'}_{=\ln a} = \boxed{a^x \ln a}$$

We've just proven that $\boxed{(a^x)' = a^x \ln a}$

(Note: The derivative is <u>NOT</u> $x a^{x-1}$ !!)

Ex: Find the derivative:

(1) $f(x) = x^2 + 5^x$

Solution: $f'(x) = \boxed{2x + 5^x \ln 5}$

(2) $f(x) = 10^x \cdot e^{2x}$

Product rule!

Solution: $f'(x) = (10^x)' e^{2x} + 10^x (e^{2x})'$

chain rule.

$= \boxed{10^x \ln 10 \cdot e^{2x} + 10^x \cdot e^{2x} \cdot 2}$

(3) $f(x) = \dfrac{x^3}{2^x+1}$

Quotient rule!

Solution: $f'(x) = \dfrac{(x^3)'(2^x+1) - x^3(2^x+1)'}{(2^x+1)^2}$

$= \boxed{\dfrac{3x^2(2^x+1) - x^3 \cdot 2^x \ln 2}{(2^x+1)^2}}$

(4) $f(x) = 7^{3x^2-5}$

chain rule!

Solution: $f'(x) = 7^{3x^2-5} \ln 7 \cdot (3x^2-5)'$

$= \boxed{7^{3x^2-5} \ln 7 \cdot 6x}$

As before the chain rule implies that

$$\boxed{\left(a^{f(x)}\right)' = a^{f(x)} \ln a \cdot f'(a)}$$

Ex: If $f(x) = x \cdot 3^{x^2}$, then

$$f'(x) = (x)' \cdot 3^{x^2} + x \cdot (3^{x^2})' \qquad \text{(Product rule)}$$

$$= \boxed{3^{x^2} + x \cdot 3^{x^2} \cdot \ln 3 \cdot (2x)}$$

Ex: If $f(x) = \dfrac{6^{\sqrt{x}}}{5x^2+2}$, then

$$f'(x) = \frac{(6^{\sqrt{x}})'(5x^2+2) - 6^{\sqrt{x}}(5x^2+2)'}{(5x^2+2)^2} \qquad \text{(Quotient rule)}$$

$$= \boxed{\frac{6^{\sqrt{x}} \cdot \ln 6 \cdot \frac{1}{2}x^{-1/2} \cdot (5x^2+2) - 6^{\sqrt{x}} \cdot (10x)}{(5x^2+2)^2}}$$

Ex: Find the equation of the tangent line to $f(x) = xe^x$ at $x=0$.

Solution: Slope: $f'(x) = (x)'e^x + x(e^x)'$ (Product rule)

$$= e^x + xe^x$$

So $f'(0) = e^0 + 0 \cdot e^0 = 1$.

A point on our line $y = x + b$ is $(0, f(0)) = (0, 0)$.

Thus, $0 = 0 + b \Rightarrow b = 0$.

The tangent line at $x=0$ is $\boxed{y = x}$.