§6.3 – The Chain Rule

We can handle derivatives of products and quotients.
What about compositions?

Ex:

	$f(x)$	$g(x)$	$f(g(x))$
(1)	x^{100}	$2x^2-5x+1$	$(2x^2-5x+1)^{100}$
(2)	$\frac{1}{x^{29}}$ $(=x^{-29})$	$x^3-\sqrt{x}$	$\frac{1}{(x^3-\sqrt{x})^{29}}$
(3)	$\sqrt{x}$	x^5+7x-4	$\sqrt{x^5+7x-4}$

We know the derivative of each $f(x)$ (outside function)...

(1) $f'(x) = 100x^{99}$

(2) $f'(x) = -29x^{-30}$

(3) $f'(x) = \frac{1}{2}x^{-1/2}$

and we know the derivative of each $g(x)$ (inside function).

(1) $g'(x) = 4x-5$

(2) $g'(x) = 3x^2 - \frac{1}{2}x^{-1/2}$

(3) $g'(x) = 5x^4+7$

Can we use this to find $(f(g(x)))'$? Yes!
We can use the chain rule.

<u>The Chain Rule</u>: The derivative of $f(g(x))$ is
$$\underbrace{f'(g(x))}_{\text{Derivative of outside with inside left alone.}} \cdot \underbrace{g'(x)}_{\text{Derivative of inside}}.$$

Derivative of inside

Derivative of outside with inside left alone.

<u>Ex</u>: If $y = (2x^2-5x+1)^{100}$, then

$$y' = 100(2x^2-5x+1)^{99} \cdot (2x^2-5x+1)'$$
$$= \boxed{100(2x^2-5x+1)^{99} \cdot (4x-5)}$$

<u>Ex</u>: If $y = \dfrac{1}{(x^3-\sqrt{x})^{29}}$ $\left(= (x^3-\sqrt{x})^{-29}\right)$, then

$$y' = -29(x^3-\sqrt{x})^{-30} \cdot (x^3-\sqrt{x})'$$
$$= \boxed{-29(x^3-\sqrt{x})^{-30} \cdot (3x^2-\tfrac{1}{2}x^{-\frac{1}{2}})}$$

<u>Ex</u>: If $y = \sqrt{x^5+7x-4}$, then

$$y' = \tfrac{1}{2}(x^5+7x-4)^{-1/2} \cdot (x^5+7x-4)'$$
$$= \boxed{\tfrac{1}{2}(x^5+7x-4)^{-1/2} \cdot (5x^4+7)}$$

A special case of the chain rule is the following

<u>The Generalized Power Rule</u>: The derivative of $(f(x))^n$ is
$$n(f(x))^{n-1} \cdot f'(x)$$

Ex: If $y=(-x+x^4)^{-5/3}$, then

$$y' = -\frac{5}{3}(-x+x^4)^{-8/3}\cdot(-x+x^4)'$$

$$= \boxed{-\frac{5}{3}(-x+x^4)^{-8/3}\cdot(-1+4x^3)}$$

Let's recap what we know.

Rule	Function	Derivative
(Constant Rule)	K (Constant)	0
(Constant · Function)	$c\cdot f(x)$	$c\cdot f'(x)$
(Power Rule)	x^n	nx^{n-1}
(Sum/Difference)	$f(x)\pm g(x)$	$f'(x)\pm g'(x)$
(Product Rule)	$f(x)\cdot g(x)$	$f'(x)g(x)+f(x)g'(x)$
(Quotient Rule)	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
(Chain Rule)	$f(g(x))$	$f'(g(x))\cdot g'(x)$
(Gen. Power Rule)	$(f(x))^n$	$n(f(x))^{n-1}\cdot f'(x)$

Ex: Use any method to find the derivative:

(1) $f(x) = 6x(3x-5)^{10}$

Solution: Product of $6x$ and $(3x-5)^{10}$. Use product rule!

$$f'(x) = (6x)'(3x-5)^{10} + 6x\Big(\underbrace{(3x-5)^{10}}_{\text{chain rule!}}\Big)'$$

$$= 6(3x-5)^{10} + 6x \cdot 10(3x-5)^9 \cdot (3x-5)'$$

$$= 6(3x-5)^{10} + 60x(3x-5)^9 \cdot 3$$

$$= \boxed{6(3x-5)^{10} + 180x(3x-5)^9}$$

(2) $f(x) = \dfrac{(2x+3)^3}{4x^2-1}$

Solution: Let's start with quotient rule.

$$f'(x) = \frac{\big((2x+3)^3\big)'(4x^2-1) - (2x+3)^3(4x^2-1)'}{(4x^2-1)^2}$$

chain rule!

$$= \frac{3(2x+3)^2 \cdot (2x+3)'(4x^2-1) - (2x+3)^3(8x)}{(4x^2-1)^2}$$

$$= \boxed{\frac{6(2x+3)^2(4x^2-1) - 8x(2x+3)^3}{(4x^2-1)^2}}$$

(3) $f(x) = \big((x+1)^2+5\big)^3$

Solution: This is a function composition (inside $= (x+1)^2+5$).
By the chain rule,

$$f'(x) = 3\big((x+1)^2+5\big)^2 \cdot \big((x+1)^2+5\big)'$$

$$= 3\big((x+1)^2+5\big)^2 \cdot \big((x+1)^2\big)' \quad \text{(chain rule again!)}$$

$$= 3\big((x+1)^2+5\big)^2 \cdot 2(x+1) \cdot \underbrace{(x+1)'}_{=1} = \boxed{6\big((x+1)^2+5\big)^2 \cdot (x+1)}$$

Ex: Suppose f and g are functions with $f(2)=3$ $\quad f'(2)=1$

$g(2)=7$ $\quad g'(2)=2$

If $h(x)=\dfrac{(f(x))^2}{g(x)}$, what is $h'(2)$?

Solution: We'll use the quotient rule to find $h'(x)$:

$$h'(x)=\frac{[(f(x))^2]'g(x)-(f(x))^2g'(x)}{(g(x))^2}$$

chain rule!

$$=\frac{2f(x)\cdot f'(x)\cdot g(x)-(f(x))^2g'(x)}{(g(x))^2}$$

Now, $h'(2)=\dfrac{2f(2)\cdot f'(2)\cdot g(2)-(f(2))^2g'(2)}{(g(2))^2}$

$$=\frac{2(3)\cdot(1)\cdot(7)-(3)^2\cdot(2)}{(7)^2}$$

$$=\frac{42-18}{49}=\boxed{\frac{24}{49}}$$

Ex: Find the equation of the tangent line to $f(x)=(2x^3+x)^3$ at $x=1$.

Solution: $f'(x)=3(2x^3+x)^2\cdot(2x^3+x)'$

$=3(2x^3+x)^2\cdot(6x^2+1)$

So the slope at $x=1$ is $f'(1)=3(2(1)^3+1)^2(6(1)^2+1)$

$=3\cdot 9\cdot 7=189$

Our line is $y=189x+b$. Using $(1,f(1))=(1,27)$, we get

$27=189(1)+b \Rightarrow b=27-189=-162$

So, $\boxed{y=189x-162.}$