§6.2 - Derivatives of Products and Quotients.

Q: Suppose we have a product $y = f(x) \cdot g(x)$. Is the derivative $y' = f'(x) \cdot g'(x)$?

A: Nooooo!!

Ex: If $f(x) = x$, $g(x) = x+1$, then $f'(x) = 1$ and $g'(x) = 1$, so $f'(x) \cdot g'(x) = 1$.

However, $y = f(x)g(x) = x(x+1) = x^2 + x$, so $y' = 2x + 1$ (Not 1!)

New Q: If $y = f(x) \cdot g(x)$, is there a simple formula for y'?

New A: Let's see...

$$y' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add and subtract $f(x)g(x+h)$ (Dirty trick, I know...)

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x)g(x+h) + f(x)g(x+h)} - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right] g(x+h) + \left[\frac{g(x+h) - g(x)}{h}\right] f(x+h)$$

$$= f'(x)g(x) + g'(x)f(x).$$

Congratulations, you've just proven the product rule!

$\boxed{\text{The Product Rule: The derivative of } f(x)\cdot g(x) \text{ is } f'(x)g(x) + f(x)g'(x).}$

Ex: Find the derivative.

(1) $f(x) = (7x-1)(3x^2+x)$.

Solution: By the product rule, we have

$$\begin{aligned} f'(x) &= (7x-1)'(3x^2+x) + (7x-1)(3x^2+x)' \\ &= 7(3x^2+x) + (7x-1)(6x+1) \\ &= (21x^2+7x) + (42x^2+x-1) \\ &= \boxed{63x^2+8x-1} \end{aligned}$$

(2) $f(x) = (\sqrt{x}+1)(2x^2-x+1)$

Solution: Product rule again!

$$\begin{aligned} f'(x) &= (\sqrt{x}+1)'(2x^2-x+1) + (\sqrt{x}+1)(2x^2-x+1)' \\ &= \left(\tfrac{1}{2}x^{-1/2}\right)(2x^2-x+1) + (x^{1/2}+1)(4x-1) \\ &= \left(x^{3/2} - \tfrac{1}{2}x^{1/2} + \tfrac{1}{2}x^{-1/2}\right) + \left(4x^{3/2} - x^{1/2} + 4x - 1\right) \\ &= \boxed{5x^{3/2} + 4x - \tfrac{3}{2}x^{1/2} + \tfrac{1}{2}x^{-1/2} - 1.} \end{aligned}$$

What about something like $f(x) = \dfrac{2x-1}{6-x}$?

We need a new rule for quotients!

The Quotient Rule: The derivative of $f(x)/g(x)$ is
$$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

This looks a bit more complicated, but it can be remembered using a cute rhyme(†):

"Low d(high) minus high d(low),
Square the bottom and away we go!"

Note: The rhyme says nothing about the derivative being $\frac{f'(x)}{g'(x)}$, because this is FALSE!

Ex: Find the derivative.

(1) $f(x) = \frac{2x-1}{6-x}$

Solution: We'll use the quotient rule:

$$f'(x) = \frac{(2x-1)'(6-x) - (2x-1)(6-x)'}{(6-x)^2}$$
$$= \frac{(2)(6-x) - (2x-1)(-1)}{(6-x)^2}$$
$$= \frac{(12-2x) - (-2x+1)}{(6-x)^2}$$
$$= \boxed{\frac{11}{(6-x)^2}}$$

(†) cute rhyme is best appreciated live – come to class!

(2) $f(x) = \dfrac{x^2+5x+1}{3x+2}$

Solution: Quotient rule!

$$f'(x) = \frac{(x^2+5x+1)'(3x+2) - (x^2+5x+1)(3x+2)'}{(3x+2)^2}$$

$$= \frac{(2x+5)(3x+2) - (x^2+5x+1)(3)}{(3x+2)^2}$$

$$= \frac{(6x^2+19x+10) - (3x^2+15x+3)}{(3x+2)^2}$$

$$= \boxed{\frac{3x^2+4x+7}{(3x+2)^2}}$$

Ex: Find the equation of the tangent line to $f(x)$.

(1) $f(x) = \dfrac{9-7x}{1-x}$ at $x=5$

Solution: By quotient rule,

$$f'(x) = \frac{(9-7x)'(1-x) - (9-7x)(1-x)'}{(1-x)^2}$$

$$= \frac{(-7)(1-x) - (9-7x)(-1)}{(1-x)^2}$$

$$= \frac{(7x-7)+(9-7x)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

At $x=5$, $f'(5) = \dfrac{2}{(1-5)^2} = \dfrac{2}{16} = \dfrac{1}{8}$.

Our line is $y = \frac{1}{8}x + b$, and using the point $(5, f(5)) = \left(5, \frac{13}{2}\right)$, we get

$$\frac{13}{2} = \frac{1}{8}(5) + b \Rightarrow b = \frac{13}{2} - \frac{5}{8} = \frac{52-5}{8} = \frac{47}{8}$$

So, our line is $\boxed{y = \frac{1}{8}x + \frac{47}{8}}$

(2) $f(x) = \frac{3x^2-1}{2\sqrt{x}}$ at $x=1$.

<u>Solution:</u> $f'(x) = \frac{(3x^2-1)'(2\sqrt{x}) - (3x^2-1)(2\sqrt{x})'}{(2\sqrt{x})^2}$

$$= \frac{(6x)(2\sqrt{x}) - (3x^2-1)\left(2\cdot\frac{1}{2}x^{-1/2}\right)}{4x}$$

$$= \frac{12x^{3/2} - 3x^{3/2} + x^{-1/2}}{4x}$$

$$= \frac{9x^{3/2} + x^{-1/2}}{4x}$$

At $x = 1$, $f'(1) = \frac{9(1)^{3/2} + (1)^{-1/2}}{4} = \frac{10}{4} = \frac{5}{2}$

Our line is $y = \frac{5}{2}x + b$.

Using the point $(1, f(1)) = (1, 1)$, we have

$$1 = 5/2\,(1) + b \Rightarrow b = 1 - 5/2 = -3/2$$

Thus, the tangent line is $\boxed{y = 5/2\,x - 3/2}$