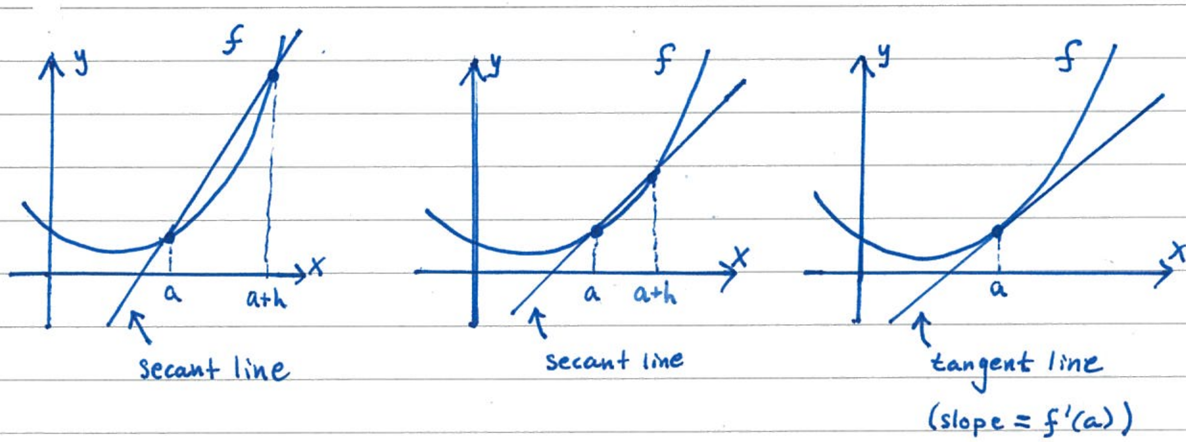
§5.4 - Definition of the Derivative

The instantaneous rate of change of $f(x)$ at $x=a$,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (a \in \mathbb{R})$$

is called the derivative of $f(x)$ at $x=a$, and is denoted by $f'(a)$.

It is also the slope of the tangent line at $x=a$. This is the line that touches $f(x)$ at $x=a$ and at no other nearby points.



We can think of the derivative as a function in its own right:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

(This is basically what we have above, except now x is a variable instead of some $a \in \mathbb{R}$)

Note: The derivative $f'(x)$ is sometimes denoted $\frac{d}{dx}\left(f(x)\right)$.

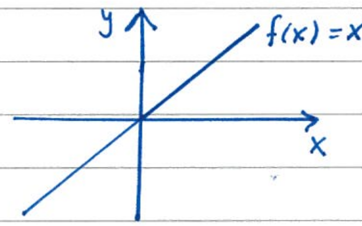
Ex: Find the derivative of $f(x)$ <u>using the definition</u>.

(Later we will learn faster methods to compute $f'(x)$, but if you are asked to use the definition, you <u>must</u> do it this way!!)

(1) $f(x) = x$

Solution: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h}$$
$$= \lim_{h \to 0} \frac{h}{h} = \boxed{1}$$

($f'(x)$ is constant!)



Is this surprising? It shouldn't be. Of course any tangent line for $f(x) = x$ should have slope 1.

(2) $f(x) = x^2$

Solution: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2xh + h^2) - \cancel{x^2}}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} 2x + h = \boxed{2x}$$

(3) $f(x) = x^3$.

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 \quad = \boxed{3x^2}$$

[What do you think $f'(x)$ is for $f(x) = x^4$? Maybe $4x^3$? Give it a try!]

(4) $f(x) = \sqrt{x}$

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad = \boxed{\frac{1}{2\sqrt{x}}}$$

Ex: What is the derivative of $f(x) = |x|$ at $x = 0$?

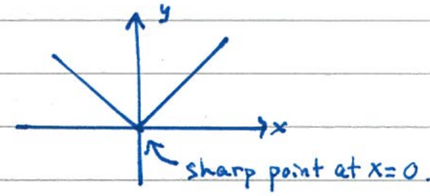
Solution: Let's try using the definition!

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

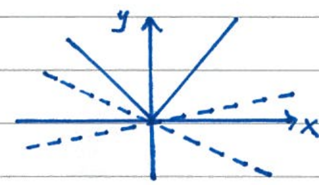
$$= \lim_{h\to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} \quad \boxed{\text{DNE}} \quad (\text{see } \S 5.1)$$

Uhh... wat?! It doesn't exist??
This is less surprising when we look at the graph of $f(x) = |x|$.



See, the derivative doesn't exist at sharp points because there are several possible tangent lines!



Ex: Find the equation of the tangent line to $f(x) = x^2 + 3x + 2$ at $x = 2$.

Solution: Recall that a line has equation

$$y = mx + b.$$

y-intercept.

slope, given by $f'(2)$.

$$m = f'(2) = \lim_{h\to 0} \frac{[(2+h)^2 + 3(2+h) + 2] - [2^2 + 3(2) + 2]}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{4} + 4h + h^2) + (\cancel{6} + 3h) + \cancel{2} - \cancel{12}}{h}$$

$$= \lim_{h\to 0} \frac{7h + h^2}{h} = \lim_{h\to 0} 7 + h = 7$$

A point on the line $y = 7x + b$ is $(2, f(2)) = (2, 12)$,
so $12 = 7(2) + b \Rightarrow b = 12 - 14 = -2$.

Tangent line: $\boxed{y = 7x - 2}$