§5.3 – Rates of Change

While driving down the 401, every 30 minutes you record how far you've travelled:

Time	0 hr	0.5 hr	1 hr	1.5 hr	2 hr	2.5 hr	3 hr
Distance	0 km	55 km	100 km	130 km	200 km	250 km	300 km

Average speed in those 3 hours was

$$\frac{\text{distance}}{\text{time}} = \frac{300\text{ km}}{3\text{ hr}} = \boxed{100\text{ km/hr.}}$$

What was your average speed in the first 1.5 hours?

$$\frac{\text{distance}}{\text{time}} = \frac{130\text{ km}}{1.5\text{ hr}} \approx \boxed{86.67\text{ km/hr}}$$

What was your average speed in the last 1.5 hours?

$$\frac{\text{distance}}{\text{time}} = \frac{300\text{ km} - 130\text{ km}}{1.5\text{ hr}} = \frac{170\text{ km}}{1.5\text{ hr}} \approx \boxed{113.33\text{ km/hr}}$$

Get the idea? Given $f(x)$, the average rate of change between $x = a$ and $x = b$ is

$$\boxed{\frac{f(b) - f(a)}{b - a}.}$$

Ex: A runner's distance (in metres) over 60 seconds is given by

$$f(t) = t + 0.1t^2.$$

(1) What is the runner's average speed over these 60 seconds?

Solution: $\dfrac{f(60)-f(0)}{60-0} = \dfrac{60+0.1(60)^2}{60} = \boxed{7\ m/s}$

(2) What is the average speed over the last 10 seconds?

Solution: $\dfrac{f(60)-f(50)}{60-50} = \dfrac{[60+0.1(60)^2]-[50+0.1(50)^2]}{10}$

$= \boxed{12\,m/s}$

(3) The last 1 second?

Solution: $\dfrac{f(60)-f(59)}{60-59} = \dfrac{[60-0.1(60)^2]-[59+0.1(59)^2]}{1}$

$\approx \boxed{12.9\ m/s}$

(4) The last millisecond? (0.001^{th} of 1 second)

Solution: $\dfrac{f(60)-f(59.999)}{60-59.999} = \ldots \approx \boxed{12.999901\ m/s}$

(5) What was the runner's instantaneous speed at $t = 60s$?

We guess 13m/s. To be sure, we need...

LIMITS!!

The <u>instantaneous rate of change</u> of $f(x)$ at $x=a$ is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Back to our example... the instantaneous speed at $t=60$ seconds is

$$\lim_{h\to 0} \frac{f(60+h)-f(60)}{h} = \lim_{h\to 0} \frac{[(60+h)+0.1(60+h)^2]-[60+0.1(60)^2]}{h}$$

$$= \lim_{h\to 0} \frac{60+h+360+12h+0.1h^2-60-360}{h}$$

$$= \lim_{h\to 0} \frac{13h+0.1h^2}{h}$$

$$= \lim_{h\to 0} 13+0.1h \quad = \boxed{13\text{ m/s}}$$

<u>Ex</u>: Find the instantaneous rate of change for $f(x)=x^2+3x$ at

(1) $x=1$

<u>Solution</u>: $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[(1+h)^2+3(1+h)]-[1^2+3(1)]}{h}$

$$= \lim_{h\to 0} \frac{(\cancel{1}+2h+h^2)+(\cancel{3}+3h)-\cancel{4}}{h}$$

$$= \lim_{h\to 0} \frac{5h+h^2}{h}$$

$$= \lim_{h\to 0} 5+h \quad = \boxed{5}$$

(2) $x = 2$

Solution: $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[(2+h)^2 + 3(2+h)] - [2^2 + 3(2)]}{h}$

$= \lim_{h \to 0} \frac{(\cancel{4} + 4h + h^2) + (\cancel{6} + 3h) - \cancel{10}}{h}$

$= \lim_{h \to 0} \frac{7h + h^2}{h}$

$= \lim_{h \to 0} 7 + h = \boxed{7}$

(3) $x = a$

Solution: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[(a+h)^2 + 3(a+h)] - [a^2 + 3a]}{h}$

$= \lim_{h \to 0} \frac{(\cancel{a^2} + 2ah + h^2) + (\cancel{3a} + 3h) - \cancel{a^2} - \cancel{3a}}{h}$

$= \lim_{h \to 0} \frac{2ah + 3h + h^2}{h}$

$= \lim_{h \to 0} 2a + 3 + h = \boxed{2a + 3}$

Congratulations! You've just computed your first <u>derivative</u>!