§5.2 - Continuity

Intuition: A function is <u>continuous</u> if you can draw its graph without lifting your pen.

Ex: Polynomials, logarithms, exponentials, $\sin x$, $\cos x$ are all continuous.

Formally: A function $f(x)$ is <u>continuous at $x=a$</u> if

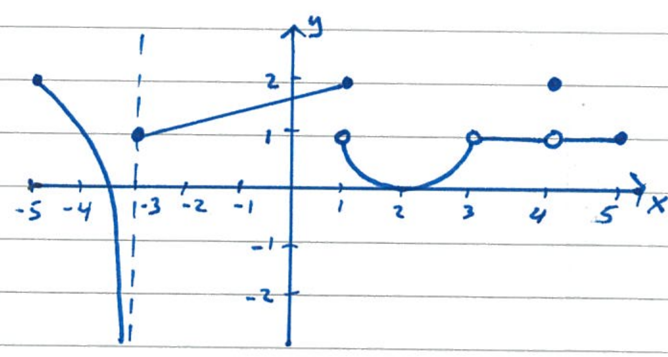
1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists, and
3. $\lim_{x\to a} f(x) = f(a)$

(we often just write 3.)

If $f(x)$ is continuous at every point in an interval I, we say that $f(x)$ is <u>continuous on I</u>.

Ex: Where is this function continuous?

Where is it discontinuous?



It is discontinuous at...

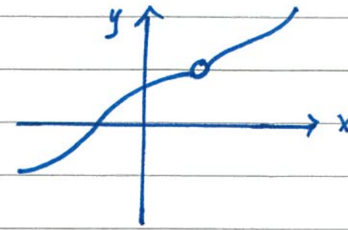
- $x=-3$ (limit DNE)
- $x=1$ (limit DNE)
- $x=3$ ($f(3)$ DNE)
- $x=4$ (limit exists, but doesn't equal $f(4)$!)

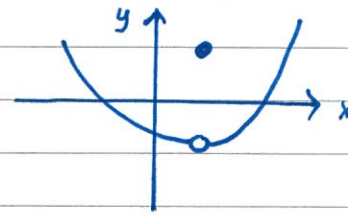
It is continuous everywhere else in $[-5,5]$.

There are <u>3 types of discontinuity</u>.

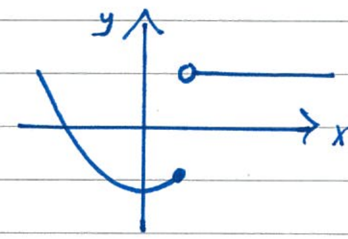
1. If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, it is a <u>removable discontinuity</u>.

   (So either $f(a)$ DNE or it is just different from $\lim_{x \to a} f(x)$)

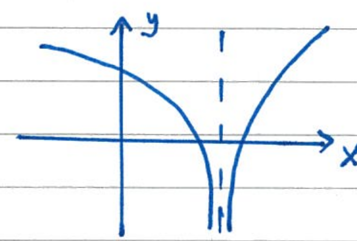




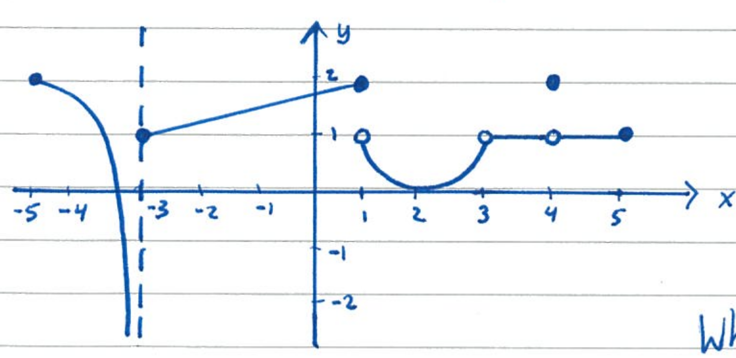
2. If both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are finite, but not equal, it is a <u>jump discontinuity</u>.



3. If $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$ it is an <u>infinite discontinuity</u>.



Refer back to the function from the 1^{st} example.



What types of discontinuities are present?

Solution:

- $\lim_{x \to -3^-} f(x) = -\infty \Rightarrow$ infinite discontinuity at $x = -3$.
- $\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = 1$. The limits are finite but not equal $\Rightarrow$ jump discontinuity at $x = 1$.
- $\lim_{x \to 3^-} f(x) = 1$ and $\lim_{x \to 3^+} f(x) = 1$ (equal) but $f(1)$ DNE.

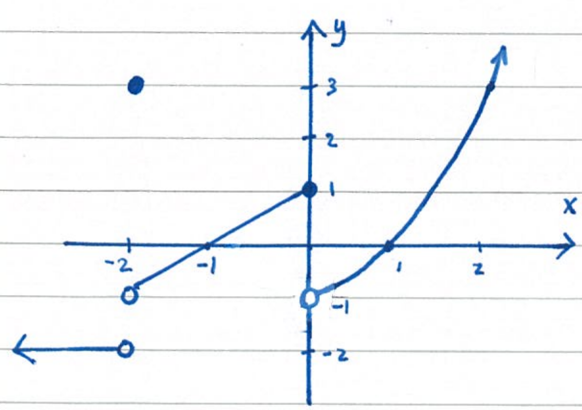
  $\Rightarrow$ removable discontinuity at $x = 3$.
- $\lim_{x \to 4^-} f(x) = 1$ and $\lim_{x \to 4^+} f(x) = 1$ (equal) but $f(1) \neq 1$.

  $\Rightarrow$ removable discontinuity at $x = 4$.

Ex: Sketch $f(x) = \begin{cases} -2 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ x+1 & \text{if } -2 < x \leq 0 \\ x^2 - 1 & \text{if } x > 0 \end{cases}$

Where is $f(x)$ continuous? What types of discontinuities does $f(x)$ possess?

Solution:



→ $\lim_{x \to -2^-} f(x) = -2$, $\lim_{x \to -2^+} f(x) = -1$

Finite but not equal, so jump discontinuity at $x = -2$.

→ $\lim_{x \to 0^-} f(x) = 1$, $\lim_{x \to 0^+} f(x) = -1$

Finite but not equal, so jump discontinuity at $x = 0$.

→ Elsewhere, $f(x)$ is continuous.