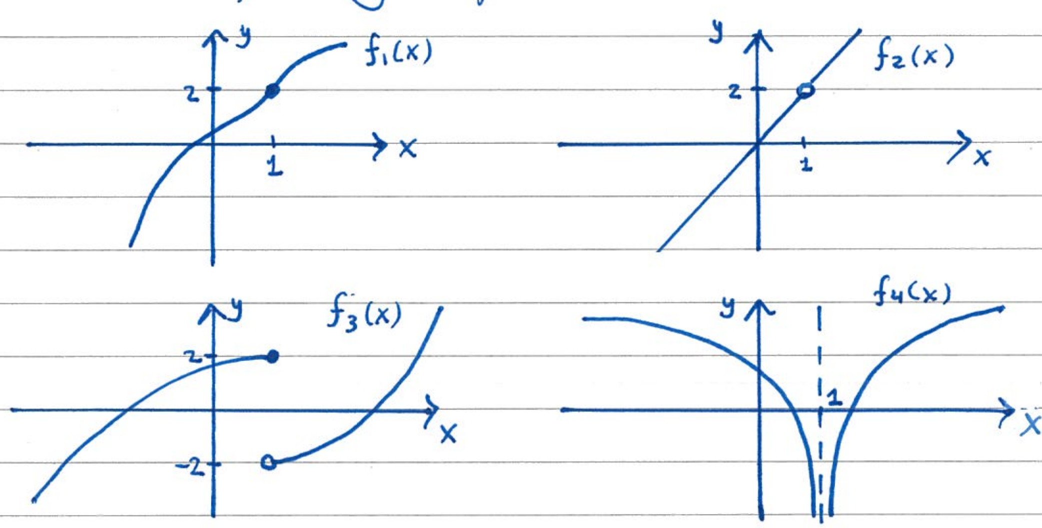
Chapter 5 - Calculus Begins (FINALLY!)

3 main topics
- **Limits** (What does $f(x)$ do as x approaches a certain value?)
- **Continuity** (Is $f(x)$ well-behaved or does it jump around?
- **Derivatives** (How "steep" is $f(x)$ at a point x?)

§5.1 - Limits

Consider the following 4 functions:



Note: f_1 and f_3 are defined at $x=1$, while f_2 and f_4 are not.

Nevertheless, we can still talk about the behaviour of each function as x gets <u>infinitely close</u> to 1!

As $x \to 1$ ...
- $f_1(x)$ approaches 2
- $f_2(x)$ approaches 2
- $f_3(x)$ approaches 2 if x comes from the left
  -2 if x comes from the right
- $f_4(x)$ approaches $-\infty$.

If $f(x)$ approaches a finite number L as x gets infinitely close to a but not equal to a, we say

"the <u>limit</u> as x approaches a of $f(x)$ is L".

and write $\boxed{\lim_{x \to a} f(x) = L}$

<u>Note:</u> L must be the same if x comes from the right or the left!

These limits are denoted by

$\lim_{x \to a^-} f(x)$	($x \to a$ from left)
$\lim_{x \to a^+} f(x)$	($x \to a$ from right)

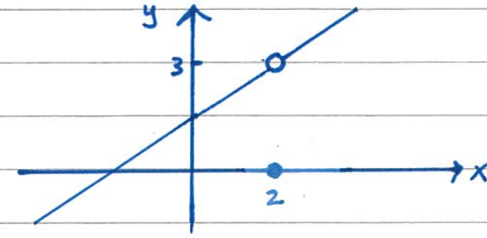
If $f(x)$ does not approach a finite value, or if the left / right limits are different, we say

"the limit as x approaches a of $f(x)$ <u>does not exist</u>" (DNE).

<u>Ex:</u> Let f_1, f_2, f_3, f_4 be as before.

- $\lim_{x \to 1} f_1(x) = 2$
- $\lim_{x \to 1} f_2(x) = 2$
- $\lim_{x \to 1^-} f_3(x) = 2$ while $\lim_{x \to 1^+} f_3(x) = -2 \Rightarrow \lim_{x \to 1} f_3(x)$ DNE
- $\lim_{x \to 1} f_4(x)$ DNE ($f(x) \to -\infty$ ... NOT FINITE!)

Ex: $f(x) = \begin{cases} x+1 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$

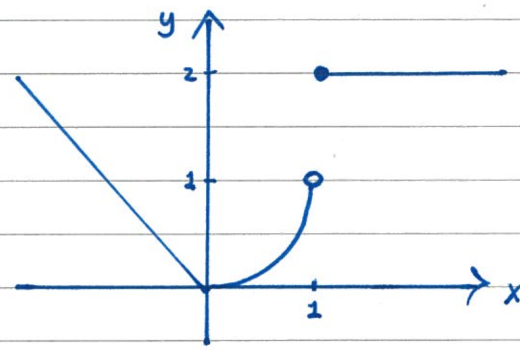


Then $f(2) = 0$ but $\lim\limits_{x \to 2} f(x) = 3$.

(we care about what happens <u>near</u> $x = 2$, not <u>at</u> $x = 2$)

Ex:

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } x \geq 1. \end{cases}$$



Then $\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} -x = 0 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x^2 = 0 \end{aligned}\right\}$ equal, so $\lim\limits_{x \to 0} f(x) = 0$.

But $\left.\begin{aligned} \lim_{\to 1^-} f(x) &= \lim_{x \to 1^-} x^2 = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} 2 = 2 \end{aligned}\right\}$ not equal, so $\lim\limits_{x \to 1} f(x)$ DNE.

We also consider <u>limits at $\pm\infty$</u> which correspond to <u>horizontal asymptotes</u> of $f(x)$.

Ex: $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$ (denominator becomes huge, so fraction becomes tiny!)

$\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$

We'll use these facts often.

Limit Rules.

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$.

1. $\lim_{x\to a} f(x) \pm g(x) = F \pm G$

2. $\lim_{x\to a} f(x) \cdot g(x) = F \cdot G$

3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{F}{G}$ (provided $G \neq 0$)

4. $\lim_{x\to a} c \cdot f(x) = c \cdot F$

5. $\lim_{x\to a} (f(x))^k = F^k$ (provided limit exists)

6. $\lim_{x\to a} b^{f(x)} = b^F$ $(b > 0)$

7. $\lim_{x\to a} \log_b(f(x)) = \log_b(F)$ $(F > 0)$

8. $\lim_{x\to a} \sin(f(x)) = \sin(F)$

   $\lim_{x\to a} \cos(f(x)) = \cos(F)$.

Remark: Rules 1., 4., and 5. imply that $\lim_{x\to a} f(x) = f(a)$ — i.e., just plug in a — when $f(x)$ is a polynomial!

Ex: $\lim_{x\to 1} x^2 - 2x + 1 = (1)^2 - 2(1) + 1 = \boxed{0}$

$\lim_{x\to 0} \sqrt{x^2+3} = \sqrt{(0)^2+3} = \boxed{\sqrt{3}}$

$\lim_{x\to \pi/2} \cos(2x) = \cos\left(2\left(\frac{\pi}{2}\right)\right) = \cos\pi = \boxed{-1}$

Ex: $\lim_{x \to 0} e^{x+1} - \ln(\sin(x) + 1) = e^{0+1} - \ln(\sin(0) + 1)$
$= e - \underbrace{\ln(1)}_{=0}$
$= \boxed{e}$

But of course the story doesn't end here...

Ex: Find the limit if it exists.

(1) $\lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5}$ (2) $\lim_{x \to \infty} \frac{3x^2 + 1}{5x^2 + 2x - 1}$

(3) $\lim_{x \to -\infty} \frac{8x + 2}{2x^2 - 5}$ (4) $\lim_{x \to \frac{\pi}{2}} \frac{\sin(2x)}{\cos x}$

(5) $\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25}$ (6) $\lim_{x \to 0} \frac{|x|}{x}$

None of these limits can be evaluated by simply "plugging in a"... more work must be done.

Strategies for Finding Limits

Step 1: Can we evaluate the limit by plugging in a?

Does it have the form $\# \cdot \infty$ $(= \infty)$, $\frac{\#}{\infty}$ $(= 0)$, or $\frac{\#}{0}$ $(= \pm\infty)$?

Step 2: Is it an indeterminate form?

$\left(0/0,\ \pm\infty/\pm\infty,\ \infty^0,\ 0^0,\ 1^\infty,\ 0 \cdot \infty,\ \infty - \infty, \text{ etc.} \right)$

Try...
- factoring and cancelling
- rationalizing denominator or numerator
- using trig. identities.
- checking left / right limits.
- factoring highest power of x in numerator and denominator.

Step 3: After each modification in step 2, try to evaluate the limit again.

Okay! Let's try (1) - (6) from previous example.

Solution:

(1)
$$\lim_{x\to 5} \frac{x^2-3x-10}{x-5} = \lim_{x\to 5} \frac{(x+2)\cancel{(x-5)}}{\cancel{x-5}}$$
$$= \lim_{x\to 5} x+2 = \boxed{7}$$

(2)
$$\lim_{x\to\infty} \frac{3x^2+1}{5x^2+2x-1} = \lim_{x\to\infty} \frac{\cancel{x^2}(3+1/x^2)}{\cancel{x^2}(5+2/x-1/x^2)}$$
$$= \lim_{x\to\infty} \frac{3+1/x^2}{5+2/x-1/x^2}$$
$$= \frac{3-0}{5+0-0} = \boxed{\frac{3}{5}}$$

(3)
$$\lim_{x\to-\infty} \frac{8x+2}{2x^2-5} = \lim_{x\to-\infty} \frac{x(8+2/x)}{x^2(2-5/x^2)}$$
$$= \lim_{x\to-\infty} \underbrace{\frac{1}{x}}_{\text{Goes to } 0} \underbrace{\left(\frac{8+2/x}{2-5/x^2}\right)}_{\text{Goes to } \frac{8+0}{2-0} = \frac{8}{2} = 4.}$$
$$= \boxed{0}$$

(4) $\displaystyle\lim_{x \to \pi/2} \frac{\sin(2x)}{\cos x} = \lim_{x \to \pi/2} \frac{2\sin x \cdot \cancel{\cos x}}{\cancel{\cos x}}$

$\displaystyle = \lim_{x \to \pi/2} 2\sin x$

$\displaystyle = 2 \cdot \sin(\pi/2) = \boxed{2}$

(5) $\displaystyle\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} = \lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} \cdot \frac{\sqrt{x} + 5}{\sqrt{x} + 5}$ (Rationalize Numerator)

$\displaystyle = \lim_{x \to 25} \frac{\cancel{x - 25}}{(\cancel{x - 25})(\sqrt{x} + 5)}$

$\displaystyle = \lim_{x \to 25} \frac{1}{\sqrt{x} + 5}$

$\displaystyle = \frac{1}{\sqrt{25} + 5} = \boxed{\frac{1}{10}}$

(6). Recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

For $\displaystyle\lim_{x \to 0} \frac{|x|}{x}$, we'll check the left & right limits.

$$\left.\begin{aligned} \lim_{x \to 0^-} \frac{|x|}{x} &= \lim_{x \to 0^-} \frac{-x}{x} = -1 \\ \lim_{x \to 0^+} \frac{|x|}{x} &= \lim_{x \to 0^+} \frac{x}{x} = 1 \end{aligned}\right\}$$

Not equal, so $\displaystyle\lim_{x \to 0} \frac{|x|}{x}$ $\boxed{\text{DNE}}$

The Squeeze Theorem

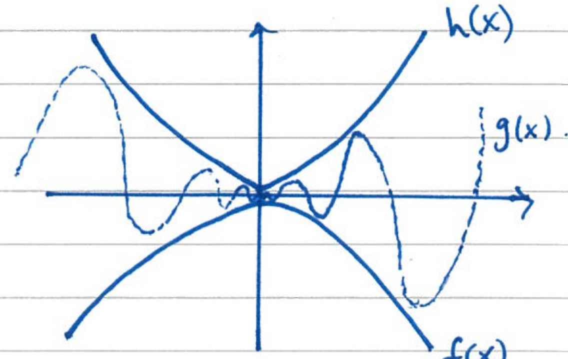
If $f(x)$, $g(x)$, $h(x)$ are functions, and

$f(x) \leq g(x) \leq h(x)$

around a, and if

$\displaystyle\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then $\displaystyle\lim_{x \to a} g(x) = L$.



Note: Good for limits involving $\sin x$ and $\cos x$, as $-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$.

Ex: Find $\lim\limits_{x \to \infty} \frac{\sin x}{x^2}$.

Solution: Since $-1 \le \sin x \le 1$, we have

$$\frac{-1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2}.$$

Thus, $$\underbrace{\lim_{x \to \infty} \frac{-1}{x^2}}_{=0} \le \lim_{x \to \infty} \frac{\sin x}{x^2} \le \underbrace{\lim_{x \to \infty} \frac{1}{x^2}}_{=0.}$$

$$\Rightarrow 0 \le \lim_{x \to \infty} \frac{\sin x}{x^2} \le 0$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x^2} = \boxed{0}$$ by the squeeze theorem.

Exercise: Find $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$