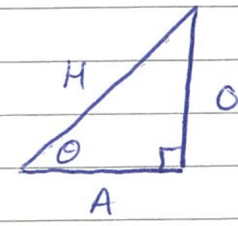
§4.4 - Trigonometric Functions

Recall: SOH CAH TOA

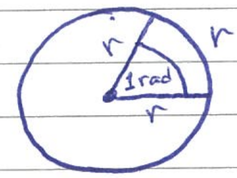
$\sin\theta = \dfrac{O}{H}$ $\quad\cos\theta = \dfrac{A}{H}$ $\quad\tan\theta = \dfrac{O}{A}$

$\csc\theta = \dfrac{1}{\sin\theta}$ $\quad\sec\theta = \dfrac{1}{\cos\theta}$ $\quad\cot\theta = \dfrac{1}{\tan\theta}$



In calculus, the angle θ is always measured in radians.

1 radian = angle that cuts off arc length equal to the radius



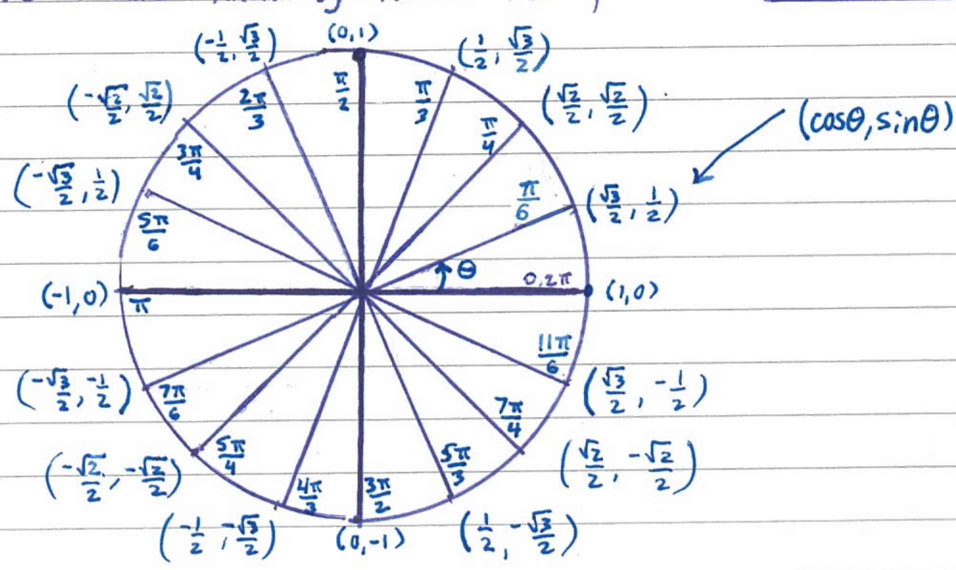
To convert degrees ↔ radians, use

$$\boxed{\text{degrees} = \frac{(\text{radians})\cdot 180}{\pi} \qquad \text{radians} = \frac{(\text{degrees})\cdot \pi}{180}}$$

Ex:

degrees	0	30	45	60	90	180	360
radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π

Most values of $\cos\theta$ and $\sin\theta$ are found with a calculator. The ones we should know by heart are found on the unit circle.



What about $\tan\theta$?

Note: $\tan\theta = \dfrac{O}{A} = \dfrac{O/H}{A/H} = \dfrac{\sin\theta}{\cos\theta}$ $\qquad\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

so if we know $\sin\theta$ and $\cos\theta$, we can calculate $\tan\theta$.

Ex: $\tan(\pi/3) = \dfrac{\sin(\pi/3)}{\cos(\pi/3)} = \dfrac{\sqrt{3}/2}{1/2} = \boxed{\sqrt{3}}$

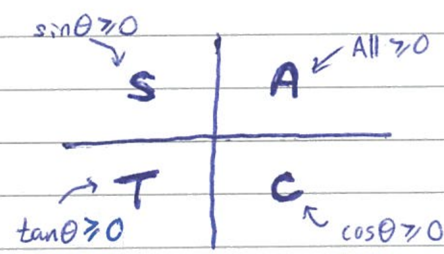
$\tan(\pi/4) = \dfrac{\sin(\pi/4)}{\cos(\pi/4)} = \dfrac{\sqrt{2}/2}{\sqrt{2}/2} = \boxed{1}$

$\tan(\pi/6) = \dfrac{\sin(\pi/6)}{\cos(\pi/6)} = \dfrac{1/2}{\sqrt{3}/2} = \boxed{\dfrac{1}{\sqrt{3}}}$

$\tan(0) = \dfrac{\sin(0)}{\cos(0)} = \dfrac{0}{1} = \boxed{0}$

Since $\cos(\pi/2) = 0$, $\tan(\pi/2)$ is <u>undefined</u>!

We can remember where $\sin\theta$, $\cos\theta$, $\tan\theta$ are + or − by using the <u>CAST rule</u>:



Trig Identities: There are lots...

Most important: $\boxed{\sin^2\theta + \cos^2\theta = 1}$ $\circledast$

Divide $\circledast$ by $\sin^2\theta$ to get $1 + \cot^2\theta = \csc^2\theta$

Divide $\circledast$ by $\cos^2\theta$ to get $\tan^2\theta + 1 = \sec^2\theta$.

Sum/Difference of angles:

$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\cos(x-y) = \cos x \cos y + \sin x \sin y$

Double angle:

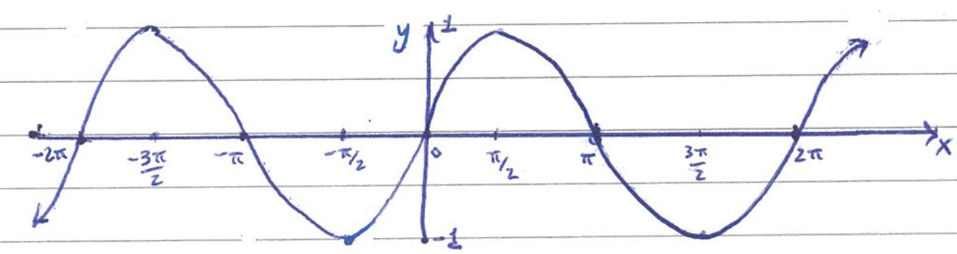
$\sin 2\theta = 2\sin\theta\cos\theta$

$$\left.\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}\right\}$$ From these we get

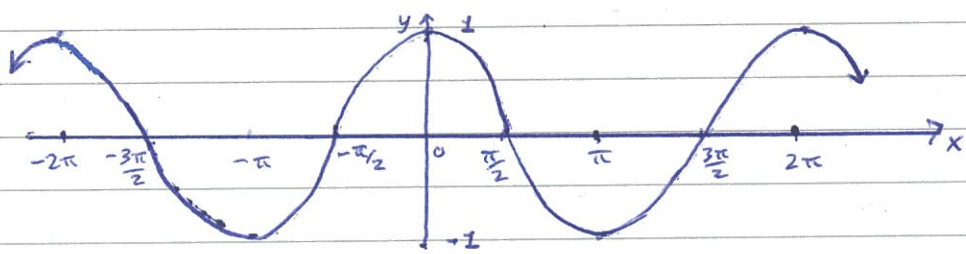
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Graphs of Trig Functions.

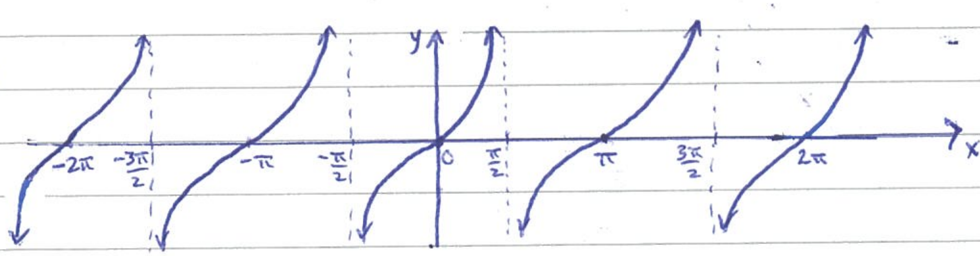
y=sinx



y=cosx



y=tanx



A <u>periodic function</u> is a function $f(x)$ such that

$$f(x) = f(x+a)$$

for some positive number a (called the <u>period</u> of f)

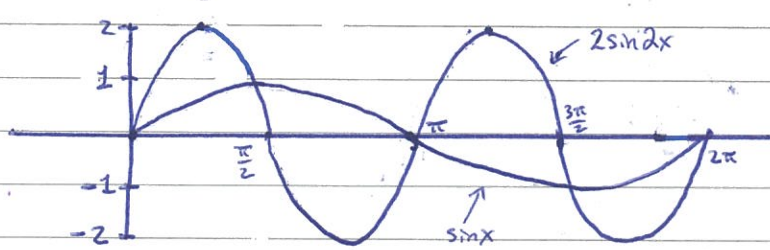
Notice from the graphs: $\sin\theta$, $\cos\theta$ are periodic with period 2π
$\tan\theta$ is periodic with period π.

We also note that $\sin\theta$ and $\cos\theta$ reach a height of 1.
This is called the <u>amplitude</u>.

In general, if $y = A\cdot\sin(B(x-C)) + D$
or $y = A\cdot\cos(B(x-C)) + D$

then A = amplitude   Period = $2\pi/B$
C = horizontal shift D = Vertical shift.

<u>Ex</u>: $y = 2\sin 2x$ has amplitude $= 2$ and period $= \frac{2\pi}{2} = \pi$
(no horizontal/vertical shifts)



<u>Solving Trig Equations</u>

<u>Ex</u>: Solve for θ: (1) $\sin 2\theta = \cos\theta$
(2) $4\cos\theta = 4 + \sin^2\theta$.

<u>Solution</u>: (1) Write $\sin 2\theta = 2\sin\theta\cos\theta$ to get

$$2\sin\theta\cos\theta = \cos\theta \Rightarrow 2\sin\theta\cos\theta - \cos\theta = 0$$
$$\Rightarrow \cos\theta\,(2\sin\theta - 1) = 0$$
$$\text{So } \cos\theta = 0$$
$$\text{OR } 2\sin\theta - 1 = 0 \quad (\text{so } \sin\theta = \tfrac{1}{2})$$

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$
$$\sin\theta = \tfrac{1}{2} \Rightarrow \theta = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6, \ldots$$

So, $$\boxed{\theta = \begin{cases} \frac{\pi}{2} + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \\ 3\pi/2 + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \\ \pi/6 + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \\ 5\pi/6 + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \end{cases}}$$

Basically, find all θ in $[0, 2\pi)$ and then account for repeats from periodicity!

(2) $$4\cos\theta = 4 + \sin^2\theta$$
$$= 4 + (1 - \cos^2\theta)$$
$$= 5 - \cos^2\theta$$

$$\Rightarrow \cos^2\theta + 4\cos\theta - 5 = 0$$
$$\Rightarrow (\cos\theta + 5)(\cos\theta - 1) = 0$$

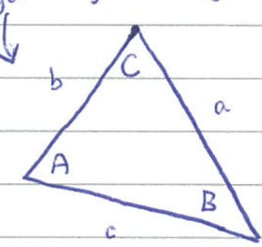
So $\cos\theta = -5$ (IMPOSSIBLE! $-1 \le \cos\theta \le 1$)

or $\cos\theta = 1$, so $\theta = 0, 2\pi, 4\pi, 6\pi, \ldots$

We have $\boxed{\theta = 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots}$

<u>Sine and Cosine Laws</u>

No longer a right triangle!

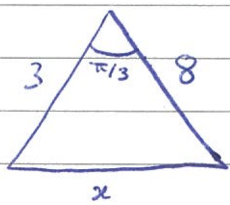


<u>Sine Law</u>: $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

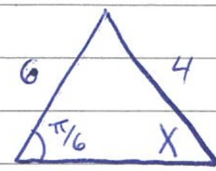
<u>Cosine Law</u>: $$c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$$

Ex: Solve for the unknown:

(1)



(2)



Solution: (1) We use cosine law:

$$x^2 = 3^2 + 8^2 - 2(3)(8)\cos(\pi/3)$$
$$= 9 + 64 - 48(1/2)$$
$$= 73 - 24$$
$$= 49$$
$$\Rightarrow \boxed{x = 7}$$

(2) We use sine law:

$$\frac{\sin X}{6} = \frac{\sin(\pi/6)}{4} \Rightarrow \sin X = \frac{6 \cdot (1/2)}{4} = \frac{3}{4}.$$
$$\Rightarrow X = \sin^{-1}(3/4)$$

Note: $\sin X$ achieves a value of $3/4$ once in quadrant I and once in quadrant II. Both are valid solutions!

With a calculator, we get

$$\boxed{\begin{array}{l} X = \sin^{-1}(3/4) \approx 0.848 \text{ radians} \\ \text{or } X = \pi - \sin^{-1}(3/4) \approx 2.29 \text{ radians} \end{array}}$$

(quadrant I solution)

(quadrant II solution)