§4.3 - Growth and Decay.

Many real life processes follow an exponential growth/decay model:

$$\boxed{A(t) = A_0 e^{k \cdot t}}$$

$A(t)$ = amount of substance at time t.
A_0 = initial amount (amount at time $t=0$)
k = constant ($k>0$ for exp. growth, $k<0$ for exp. decay)

Typical problem:
- Find k given some initial data
- Use k to find amount of substance at some other time.

Ex [Population Growth]: A rabbit colony starts with 100 rabbits. After 3 years, it has 900 rabbits. How many will it have after 6 years?

Solution: We are given $A_0 = 100$, $A(3) = 900$.
Our model is

$$A(t) = 100 e^{k \cdot t}.$$

To find $A(6)$, we should first determine k by using $A(3) = 900$.

$$\begin{aligned} A(3) = 900 &\Rightarrow 900 = 100 e^{k \cdot 3} \\ &\Rightarrow 9 = e^{3k} \\ &\Rightarrow \ln(9) = \ln(e^{3k}) = 3k \quad \text{(ln both sides)} \\ &\Rightarrow k = \frac{\ln(9)}{3} \end{aligned}$$

The model is $A(t) = 100 e^{\frac{\ln(9)}{3} \cdot t}$

Thus, $A(6) = 100e^{\frac{\ln(9)}{3} \cdot 6} = 100e^{2 \cdot \ln(9)}$

$= 100e^{\ln(9^2)}$

$= 100 \cdot 9^2$

$= \boxed{8100}$ rabbits (wow!)

Ex[Radioactive Decay]: A radioactive isotope has a half-life of 10 years. How much of this substance will be left after 23 years?

Solution: What is A_0?
We'll say $A_0 = 100$ (percent), so $A(10) = 50$
(after 10 years, only half is left!)

Our model is $A(t) = 100e^{k \cdot t}$.
Using $A(10) = 50$, we can solve for k:

$$A(10) = 50 \Rightarrow 50 = 100e^{k \cdot 10}$$
$$\Rightarrow \frac{1}{2} = e^{10k}$$
$$\Rightarrow \ln\left(\frac{1}{2}\right) = \ln(e^{10k}) = 10k \quad (\ln \text{ both sides})$$

So, $k = \frac{\ln(1/2)}{10}$.

Our model is $A(t) = 100e^{\frac{\ln(1/2)}{10} \cdot t}$.

This means that $\boxed{A(23) = 100e^{\frac{\ln(1/2)}{10} \cdot 23}} \approx 20.31$

Thus, $\approx 20.31\%$ of the substance is left after 23 years.

A slightly different example: the amount of chemical that will dissolve in a solution increases exponentially as temperature increases!

Ex [Chemical Dissolution]: At $0°C$, 1000g of chemical dissolves in a solution. At $10°C$, 1100g dissolves. At what temperature will 1500g dissolve?

Solution: We are given $\begin{cases} A_0 = 1000 \\ A(10) = 1100 \end{cases}$

Our model is $A(t) = 1000e^{kt}$ (now, t = temperature)

We'll solve for k using $A(10) = 1100$.

$$A(10) = 1100 \Rightarrow 1100 = 1000e^{k \cdot 10}$$
$$\Rightarrow \frac{11}{10} = e^{10k}$$
$$\Rightarrow \ln\left(\frac{11}{10}\right) = \ln(e^{10k}) = 10k$$
$$\text{So } k = \frac{\ln\left(\frac{11}{10}\right)}{10}$$

Our model is $A(t) = 1000e^{\frac{\ln(11/10)}{10} \cdot t}$.

We would like to know t such that $A(t) = 1500$.

Solve for t: $1500 = 1000e^{\frac{\ln(11/10)}{10} \cdot t}$
$$\Rightarrow \frac{3}{2} = e^{\frac{\ln(11/10)}{10} \cdot t}$$
$$\Rightarrow \ln(3/2) = \frac{\ln(11/10)}{10} \cdot t \Rightarrow \boxed{t = \frac{10\ln(3/2)}{\ln(11/10)}}$$
$$\approx 42.5°C$$