§4.2 - Logarithmic Functions

The logarithm function is the inverse of the exponential.

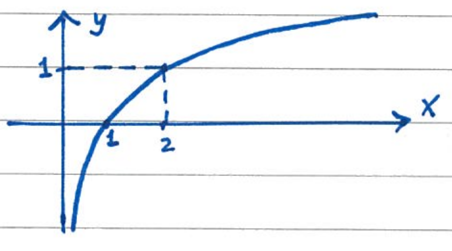
Definition: Given $a > 0$, $a \neq 1$, and $x > 0$, we write

$$\boxed{y = \log_a x \iff a^y = x}$$

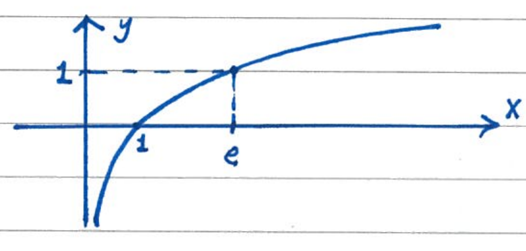
Special case: We write $\ln x$ (natural log.) in place of $\log_e x$.

The function $f(x) = \log_a x$ has
- domain $= (0, \infty)$
- range $= \mathbb{R}$
- vertical asymptote at $x = 0$ (y-axis)

Ex: $f(x) = \log_2(x)$



$g(x) = \ln x$



Properties of Logarithms

(1) $\log_a(x \cdot y) = \log_a x + \log_a y$

(2) $\log_a(x/y) = \log_a x - \log_a y$

(3) $\log(x^r) = r \cdot \log_a x$

(4) $\log_a(a^x) = x$, $a^{\log_a x} = x$ (In particular, $\ln(e^x) = x$, $e^{\ln x} = x$)

(5) $\log_a(a) = 1$, $\log_a(1) = 0$.

Ex: Simplify the following:

(1) $\log_2(6) + \log_2(15) - \log_2(5/4)$

Solution: $\log_2(6) + \log_2(15) - \log_2(5/4)$

$$= \log_2\left(\frac{6 \cdot 15}{5/4}\right)$$

$$= \boxed{\log_2(72)}$$

(2) $\ln(x^2) + \ln x$.

Solution: $\ln(x^2) + \ln x = 2\ln x + \ln x = \boxed{3\ln x}$

What if our logs have different bases, such as $\log_2(6) - \log_4(9)$?

We can use the change of basis formula!

Given $a, b, x > 0$ with $a \neq 1$, $b \neq 1$,

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

In particular,

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

We have

$$\log_2(6) - \log_4(9) = \frac{\ln 6}{\ln 2} - \frac{\ln 9}{\ln 4}$$
$$= \frac{\ln 6}{\ln 2} - \frac{\ln(3^2)}{\ln(2^2)}$$
$$= \frac{\ln 6}{\ln 2} - \frac{\cancel{2}\ln 3}{\cancel{2}\ln 2}$$
$$= \frac{\ln 6 - \ln 3}{\ln 2}$$
$$= \frac{\ln(6/3)}{\ln 2} = \frac{\ln 2}{\ln 2} = \boxed{1}$$ cool!

Solving Logarithmic Equations

Using log properties, we can solve log equations.

Ex: Solve for x.

(1) $x = \log_2(8)$

Solution: $x = \log_2(8) = \log_2(2^3) = \boxed{3}$ (Property (4))

(2) $\log_x(8/27) = 3$

Solution: $\log_x(8/27) = 3 \Leftrightarrow x^3 = 8/27$
$$\Leftrightarrow x = \sqrt[3]{\frac{8}{27}} = \boxed{\frac{2}{3}}$$

(3) $\log_4(x) = 5/2$

Solution: $\log_4(x) = 5/2 \Leftrightarrow 4^{5/2} = x$
$$\Leftrightarrow 2^5 = x$$
$$\Leftrightarrow x = \boxed{32}$$

(4) $\log_2(x) - \log_2(x-1) = 1$.

Property (2)

Solution: $\log_2(x) - \log_2(x-1) = 1 \Leftrightarrow \log_2\left(\frac{x}{x-1}\right) = 1$

$$\Leftrightarrow 2^1 = \frac{x}{x-1}$$

$$\Leftrightarrow 2x - 2 = x$$

$$\Leftrightarrow \boxed{x = 2}$$

Logs can also be used to solve exponential equations where bases are different!

Useful Fact: If $b > 0$, $b \neq 1$, and $x, y > 0$, then

$$\boxed{\log_b x = \log_b y \Leftrightarrow x = y}$$

Ex: Solve for x.

(1) $3^x = 5$

Solution: By the above fact, we can apply ln to both sides.

So $3^x = 5 \Leftrightarrow \ln(3^x) = \ln(5)$

$$\Leftrightarrow x \cdot \ln(3) = \ln(5)$$

$$\Leftrightarrow \boxed{x = \frac{\ln 5}{\ln 3}}$$

(2) $3^{2x} = 4^{x+1}$

Solution: $3^{2x} = 4^{x+1} \Leftrightarrow \ln(3^{2x}) = \ln(4^{x+1})$

$$\Leftrightarrow 2x \cdot \ln(3) = (x+1) \cdot \ln(4)$$

$$\Leftrightarrow x\,(2 \cdot \ln(3) - \ln(4)) = \ln(4)$$

So $\boxed{x = \frac{\ln(4)}{2 \cdot \ln(3) - \ln(4)}}$