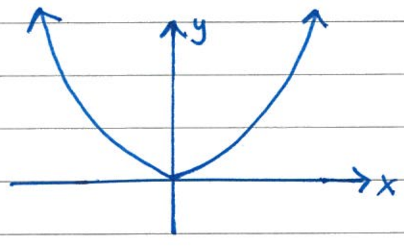
§3.5 - Polynomial and Rational Functions

Recall from Ch. 2:

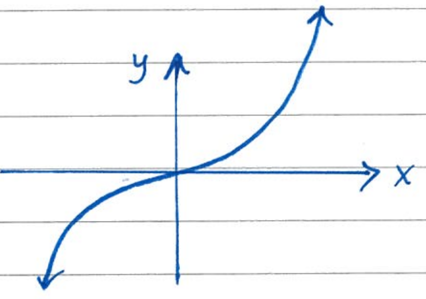
A polynomial of degree n is a function of the form

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- Domain $= \mathbb{R}$
- Range $= \mathbb{R}$ if degree n is odd



$y = x^n$ (n even)



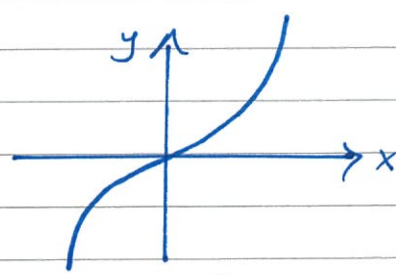
$y = x^n$ (n odd)

(roughly.

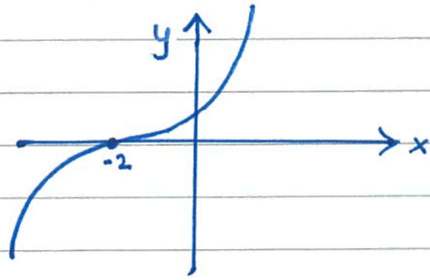
Using techniques from §3.4 we can graph some polynomials.

Ex: Starting with the graph of $y = x^3$, sketch the graph of $y = -(x+2)^3 - 1$.

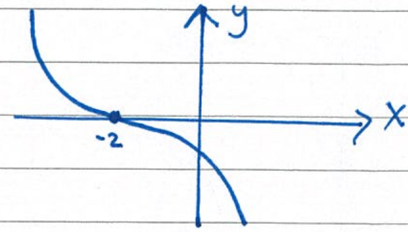
Solution: $y = x^3$ looks like



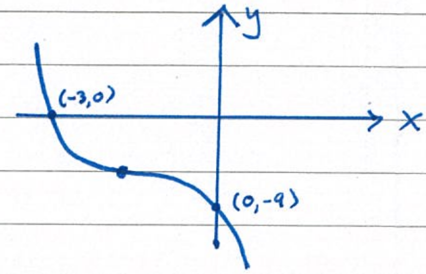
$y = (x+2)^3$ is a left translation by 2 units.



$y = -(x+2)^3$ is a reflection over the x-axis:



$y = -(x+2)^3 - 1$ is a downward shift by 1 unit:



- What's the x-intercept? Set $y = 0$.

$$0 = -(x+2)^3 - 1 \Rightarrow (x+2)^3 = -1$$
$$\Rightarrow x + 2 = -1$$
$$\Rightarrow x = -3$$

x-int is $(-3, 0)$.

- What's the y-intercept? Set $x = 0$

$$y = -(0+2)^3 - 1 = -8 - 1 = -9$$

y-int is $(0, -9)$.

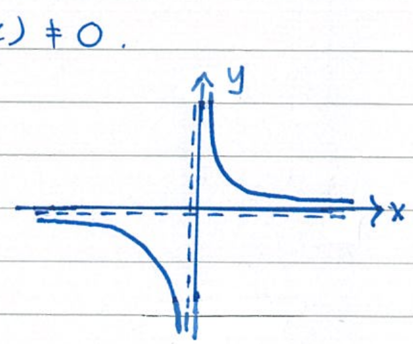
Recall: A rational function has the form

$$y = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Domain = All values of x where $q(x) \neq 0$.

Ex: $y = \frac{1}{x}$ has domain $(-\infty, 0) \cup (0, \infty)$.



The dotted lines in the graph are called asymptotes.

$y = \frac{1}{x}$ has a <u>vertical asymptote</u> at $x=0$ (y-axis) and a <u>horizontal asymptote</u> at $y=0$ (x-axis).

<u>Note:</u> If $q(a)=0$ and $p(a)\neq 0$, then the function $\frac{p(x)}{q(x)}$ has a vertical asymptote at $x=a$.

Finding horizontal asymptotes requires limits (§5.1)

<u>Ex:</u> For $y = \frac{x^2-3x+2}{x^2-3x}$ find
(a) vertical asymptotes
(b) x-intercepts
(c) y-intercept.

<u>Solution:</u> $p(x) = x^2-3x+2 = (x-2)(x-1)$

$q(x) = x^2-3x = x(x-3)$.

(a) For vertical asymptotes, we solve $q(x)=0$ to get $x=0$ or $x=3$.

Since $p(0)\neq 0$ and $p(3)\neq 0$, we have vertical asymptotes at both $\boxed{x=0 \text{ and } x=3.}$

(b) Set $y=0$

$$\frac{x^2-3x+2}{x^2-3x} = 0 \Rightarrow x^2-3x+2=0$$
$$\Rightarrow x=1 \text{ or } x=2.$$

$\therefore$ x-intercepts are $\boxed{(1,0),\ (2,0)}$

(c) Since $x=0$ is not in the domain, $\boxed{\text{no } y\text{-intercepts.}}$

Ex: Give an example of a rational function with

(i) vertical asymptotes at $x=-1$ and $x=5$, and
(ii) y-intercept at $(0,4)$.

Solution: Our rational function will be $\frac{p(x)}{q(x)}$.

Vertical asymptote $\Rightarrow q(x)=0$, so why don't we take $q(x)=(x+1)(x-5)$?

What must $p(x)$ be?

We need $\frac{p(0)}{q(0)}=4$ for the y-intercept, so
$p(0)=4\cdot q(0)=-20$.

We can take $p(x)=-20$, so

$$\boxed{y=\frac{-20}{(x+1)(x-5)}}$$ will work.

Note: $y=\frac{x-20}{(x+1)(x-5)}$ or $y=\frac{x^2+x-20}{(x+1)(x-5)}$

will also work. There are many possible solutions!