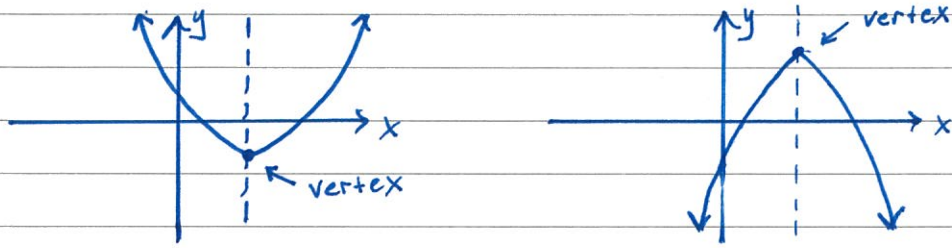
§3.4 - Quadratic Functions ; Translations & Reflections

Recall: A quadratic function has the form

$$y = ax^2 + bx + c \qquad (a, b, c \in \mathbb{R},\ a \neq 0)$$



Above: Graphs of quadratic functions, called parabolas. The "peak / trough" point is the vertex.

A parabola is symmetric about the vertical line passing through its vertex.

Completing the Square

To graph a parabola, we first write it as

$$\boxed{y = a(x-h)^2 + k}$$

- The point (h, k) is the vertex.
- The parabola opens upward $\cup$ if $a > 0$.
  opens downward $\cap$ if $a < 0$

The process of converting $ax^2 + bx + c$ to the above form is called completing the square.

3 steps in completing the square.	Ex: Complete the square for $y = 2x^2 - 4x + 3$.
(1) Factor the coefficient on x^2 from <u>all</u> x terms	$y = \underbrace{2}_{\text{Factor me!}}x^2 - 4x + 3$
You'll get something like	$\Rightarrow y = 2(x^2 - 2x) + 3$
$y = a(x^2 + px) + q$	(so $a = 2$, $p = -2$, $q = 3$)
(2) Add and subtract $\left(\frac{p}{2}\right)^2$ within the brackets.	Add and subtract $\left(\frac{p}{2}\right)^2 = 1$.
	$y = 2(x^2 - 2x + \overbrace{1 - 1}) + 3$
Remove the subtracted term from the brackets	$\Rightarrow y = 2(x^2 - 2x + 1) - 2 + 3$
	$\Rightarrow y = 2(x^2 - 2x + 1) + 1$.
(3) Factor the bracketed term as	$\frac{p}{2} = -1$, so
$\left(x + \frac{p}{2}\right)^2$	$\boxed{y = 2(x-1)^2 + 1.}$
You're done!	Yay!

Ex: Complete the square for $y = -x^2 + 4x + 5$

Solution: Just follow the 3 steps.

(1) $y = -(x^2 - 4x) + 5$ (factor coefficient)

(2) $y = -(x^2 - 4x + 4 - 4) + 5$ (add and subtract)

$\Rightarrow y = -(x^2 - 4x + 4) + 4 + 5$

(3) $y = -(x-2)^2 + 9$ (factor brackets)

Graphing a Quadratic

To graph $y = a(x-h)^2 + k$ ...

- plot vertex at (h, k)
- plot y-intercept (found by setting $x = 0$)
- plot x-intercept(s) (found by setting $y = 0$)
- Connect points to make $\cup$ if $a > 0$,
  $\cap$ if $a < 0$.

Note: The parabola may not have x-intercepts!
If $a \cdot k > 0$, then there are none.

Ex: Graph the quadratic $y = 2(x-1)^2 + 1$.

Solution: Start with the vertex.
It is at $(h, k) = (1, 1)$.

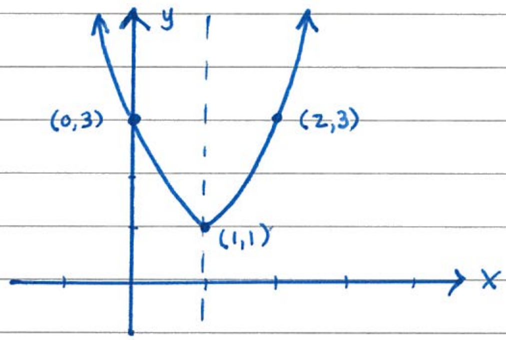
y-intercept? Set $x=0$.

$$y = 2(0-1)^2 + 1 = 2(1) + 1 = 3$$

So, y-int is $(0,3)$.

X-intercept(s)? There are none! $a \cdot K = 2 \cdot 1 = 2$ (>0)

[If we had tried setting $y=0$ and solving for x, we would get $(x-1)^2 = -1/2$. We can't square root $-\frac{1}{2}$!]



The symmetry about the dotted line shows that $(2,3)$ is also a point.

Ex: Graph the quadratic $y = -(x-2)^2 + 9$

Solution: Vertex $= (2,9)$
Since $a = -1$ (<0), parabola opens down.

y-intercept? Set $x=0$

$$y = -(0-2)^2 + 9 = -4 + 9 = 5$$

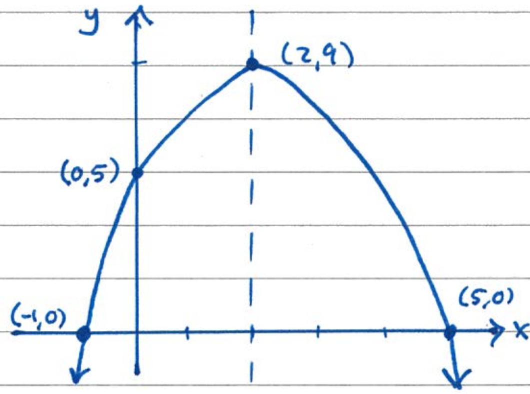
$\Rightarrow$ y-int is $(0,5)$.

X-intercept(s)? $a \cdot K = -9$, so there are x-ints!

Set $y=0$ and solve.

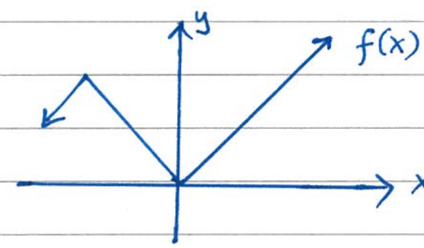
$$y = 0 \Rightarrow 0 = -(x-2)^2 + 9$$
$$\Rightarrow (x-2)^2 = 9$$
$$\Rightarrow x - 2 = \pm\sqrt{9} = \pm 3.$$

So $x = -1$ or $x = 5$. The x-ints are $(-1,0)$, $(5,0)$.



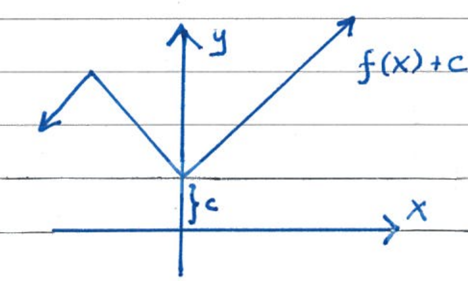
Translations & Reflections

Suppose $f(x)$ is a function with graph →

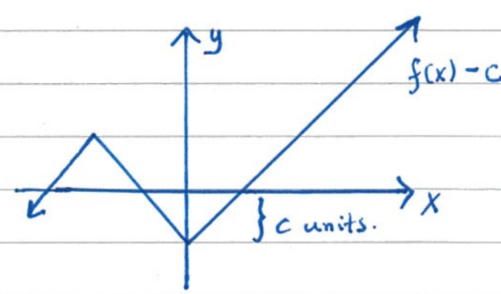


Let $c > 0$ be a constant

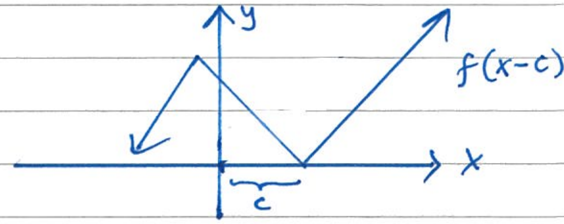
$f(x) + c$ is $f(x)$ translated up by c units.



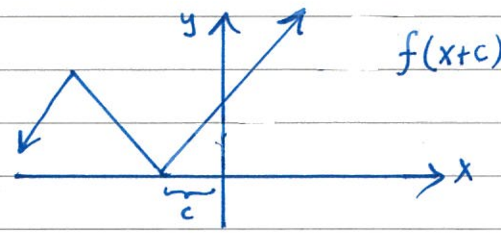
$f(x) - c$ is $f(x)$ translated down by c units.



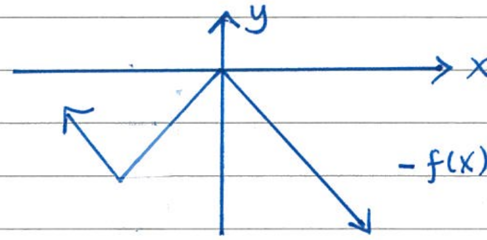
$f(x-c)$ is $f(x)$ translated right by c units



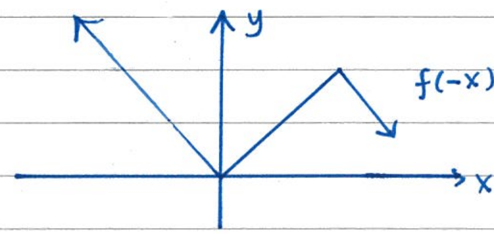
$f(x+c)$ is $f(x)$ translated left by c units



$-f(x)$ is $f(x)$ reflected over the x-axis

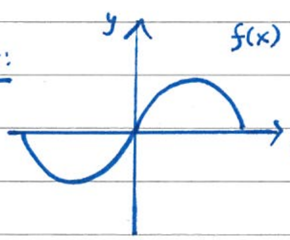


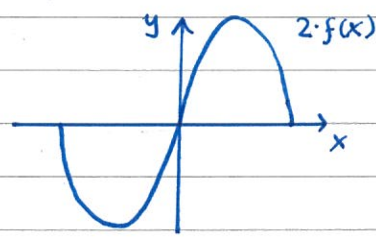
$f(-x)$ is $f(x)$ reflected over the y-axis.

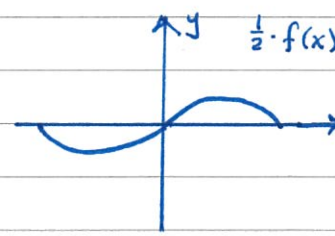


$C \cdot f(x)$ is a vertical stretch/compression

Ex:

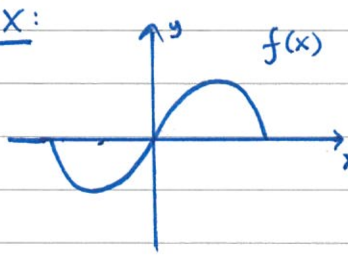


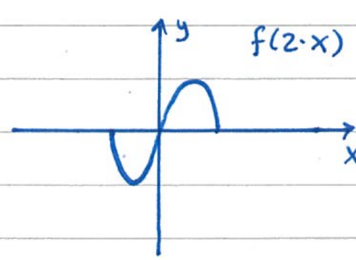


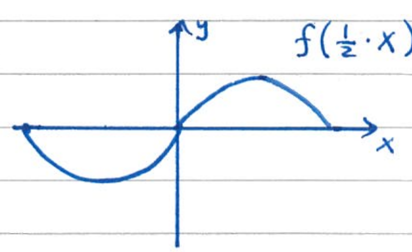


$f(c \cdot x)$ is a horizontal stretch/compression.

Ex:







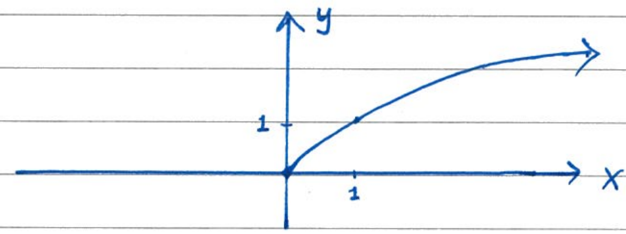
Here's the order for applying transformations

(1) horizontal stretch / compression
(2) y-axis reflection
(3) horizontal shift
(4) vertical stretch / compression
(5) x-axis reflection
(6) vertical shift.

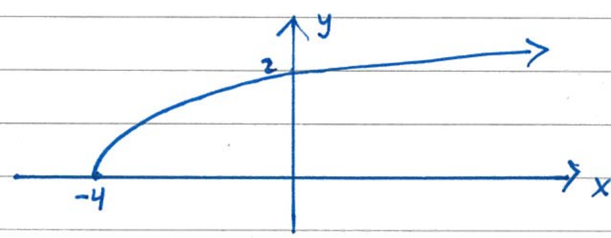
Ex: Sketch the graph of $y = -\sqrt{x+4} + 2$

Solution:

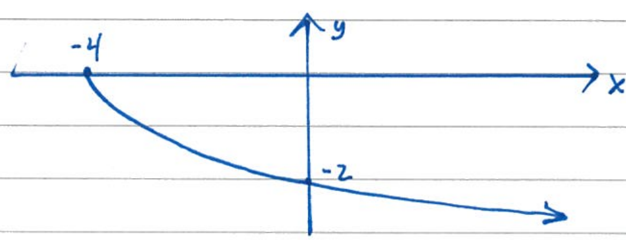
$y = \sqrt{x}$



$y = \sqrt{x+4}$
(left shift)



$y = -\sqrt{x+4}$
(x-axis reflection)



$y = -\sqrt{x+4} + 2$
(vertical shift)