§ 3.3 - Properties of Functions

A function is a rule that assigns to each element of a set exactly one element of another set.

Common notation: $y = f(x)$

y = dependent variable
x = independent variable.

Ex: The area of a circle is a function of its radius:

$$A(r) = \pi r^2$$

Ex: The volume of a cube is a function of its side length:

$$V(s) = s^3$$

The domain of a function is the set of all possible input values (x values).

The range of a function is the set of all possible output values (y values).

Ex:

Function	Domain	Range
$y = x^2$	$\mathbb{R}$ $\left(=(-\infty,\infty)\right)$	$[0,\infty)$ (can't output negatives!)
$y = \sqrt{x}$	$[0,\infty)$ (can't take $\sqrt{\ }$ of negative!)	$[0,\infty)$
$y = \cos(x)$	$\mathbb{R}$	$[-1,1]$

When finding the domain of a function...

(1) Don't allow division by 0,

(2) Don't $\sqrt{\ }$ (or even-root) a negative number,

(3) Don't take a logarithm of a number ≤ 0.

Ex: Find the domain of $f(x)$.

(1) $f(x) = \dfrac{1}{x-1}$.

Solution: Denominator is 0 when $x=1$, so this point is excluded.

$$\boxed{\text{Domain} = (-\infty, 1) \cup (1, \infty).}$$

(2) $f(x) = \sqrt{5-x}$.

Solution: If $5-x<0$, then we can't take $\sqrt{\ }$.

Note:

$$5-x \geq 0 \iff x \leq 5$$

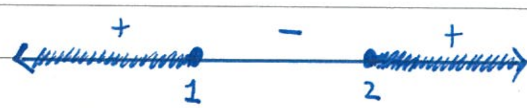
So, the domain is $\boxed{(-\infty, 5].}$

(3) $f(x) = \dfrac{\sqrt{x^2-3x+2}}{x}$

Solution: We can't divide by 0, so $x=0$ is out.

We also need $x^2-3x+2 \geq 0$ to take $\sqrt{\ }$.

We factor $x^2 - 3x + 2 = (x-2)(x-1)$ and check values around the roots:



↑ Middle interval is bad news!

Combining these restrictions, we get

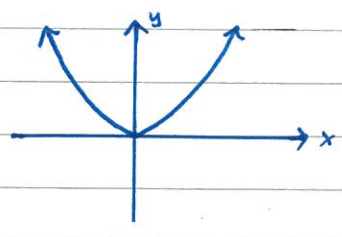
$$\boxed{\text{Domain} = (-\infty, 0) \cup (0, 1] \cup [2, \infty)}$$

Q: How do we know if an equation defines a function?

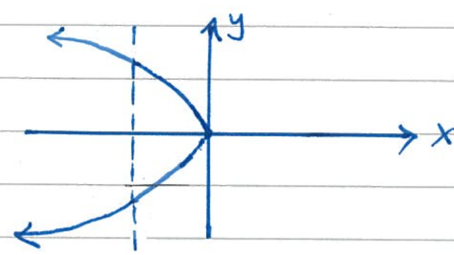
A: We use the vertical line test!

Vertical Line Test: If a vertical line passes through more than 1 point of a graph, then this is NOT the graph of a function.

Ex:



$y = x^2$ IS a function!



$x = y^2$ is NOT a function!

Evaluating Functions: To evaluate $y = f(x)$ at a point $x = a$, replace every x in $f(x)$ with a.

Ex: If $f(x) = 2x^3 + \sqrt{x}$, then

- $f(1) = 2(1)^3 + \sqrt{1}$
  $= 3$
- $f(4) = 2(4)^3 + \sqrt{4}$
  $= 2 \cdot 64 + 2 = 130$
- $f(x+h) = 2(x+h)^3 + \sqrt{x+h}$
- 

(It's a hedgehog, okay!?)

Composition of Functions

We can use functions $f(x)$ and $g(x)$ to make new functions!

The composition of f and g is the function $f \circ g$, defined by

$$(f \circ g)(x) = f(g(x))$$

(i.e., first apply g, then apply f).

We can also do the reverse composition $g \circ f$ (first apply g, then apply f).

Ex: If $f(x) = x^2 - x$ and $g(x) = 2x + 1$, then

$$\begin{aligned}(f \circ g)(x) = f(g(x)) = f(2x+1) &= (2x+1)^2 - (2x+1) \\ &= (4x^2 + 4x + 1) - (2x+1) \\ &= 4x^2 + 2x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) = g(f(x)) = g(x^2 - x) &= 2(x^2 - x) + 1 \\ &= 2x^2 - 2x + 1.\end{aligned}$$

Caution: Usually $f \circ g \neq g \circ f$!!