§1.4 - Vector Cross Product in $\mathbb{R}^3$

Given $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, how can we find $\vec{c}$ orthogonal to both $\vec{a}$ and $\vec{b}$?

Suppose
$$\vec{a} = (a_1, a_2, a_3)$$
$$\vec{b} = (b_1, b_2, b_3)$$
$$\vec{c} = (c_1, c_2, c_3)$$

We want
$$\vec{a} \cdot \vec{c} = 0 \Rightarrow a_1c_1 + a_2c_2 + a_3c_3 = 0$$
$$\vec{b} \cdot \vec{c} = 0 \Rightarrow b_1c_1 + b_2c_2 + b_3c_3 = 0$$

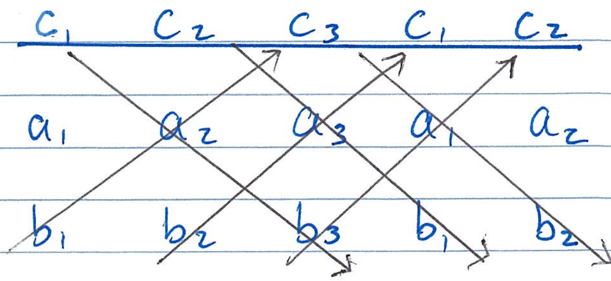
Many values of c_1, c_2, c_3 will satisfy these equations, but here's a "standard choice":

$$\vec{c} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

This is the cross product of $\vec{a}$ and $\vec{b}$:

$$\boxed{\begin{array}{l}\text{If } \vec{a}, \vec{b} \in \mathbb{R}^3, \text{ then their } \underline{\text{cross product}} \text{ is} \\ \vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)\end{array}}$$

How can we remember this mess?



For c_1, add the product on the down arrow, subtract the product on the up arrow.

Do the same for c_2, c_3.

Ex: The cross product of $\vec{a} = (-3, 1, 0)$ and $\vec{b} = (1, 1, 2)$ is

$$\vec{a} \times \vec{b} = \big((1)(2) - (0)(1),\ (0)(1) - (-3)(2),\ (-3)(1) - (1)(1)\big)$$
$$= (2 - 0,\ 0 + 6,\ -3 - 1)$$
$$= \boxed{(2, 6, -4).}$$

Sure enough, this is perpendicular to both $\vec{a}$ and $\vec{b}$.
NEAT!

Properties of Cross Product:

1. $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a}$.

2. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

3. If $\vec{a} \times \vec{b} = 0$, then either
   - ↳ $\vec{a} = \vec{0}$,
   - ↳ $\vec{b} = \vec{0}$, or
   - ↳ $\vec{a}$ is a multiple of $\vec{b}$ (ex: $\vec{a} = 4\vec{b}$)

4. $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

5. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

6. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$ for any $k \in \mathbb{R}$.

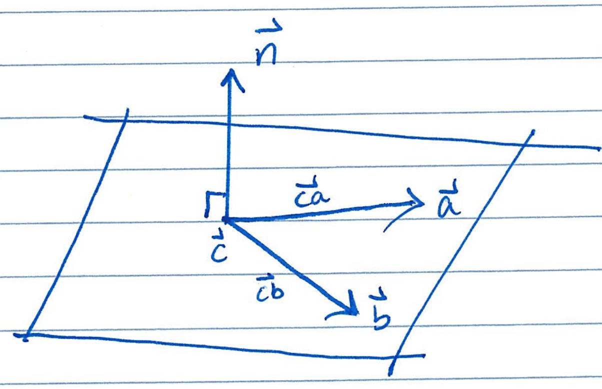
Applications

Using cross products, we can now find the equation of a plane without being handed its normal vector!

Ex: Find an equation of the plane containing
$\vec{a} = (1, -2, 1)$,
$\vec{b} = (2, -2, 1)$,
$\vec{c} = (4, 1, 1)$.

Solution: The plane must contain $\vec{ca}$ and $\vec{cb}$

$$\vec{ca} = \vec{a} - \vec{c} = (1, -2, 1) - (4, 1, 1) = (-3, -3, 0)$$
$$\vec{cb} = \vec{b} - \vec{c} = (2, -2, 1) - (4, 1, 1) = (-2, -3, 0)$$



The normal vector $\vec{n}$ is perpendicular to the plane, so it must be perpendicular to both $\vec{ca}$ and $\vec{bc}$.

We can use cross products to find $\vec{n}$!!

$$\vec{n} = (\vec{ca}) \times (\vec{cb})$$
$$= \big((-3)(0) - (0)(-3),\ (0)(-2) - (-3)(0),\ (-3)(-3) - (-3)(-2)\big)$$
$$= (0, 0, 3)$$

The equation of the plane is $0x_1 + 0x_2 + 3x_3 = d$,
So $3x_3 = d$.

To get d, plug in $\vec{a} = (1,-2,1)$ (or $\vec{b}$ or $\vec{c}$ ...)

$$3x_3 = d \Rightarrow 3(1) = d \Rightarrow d = 3.$$

The plane is $\boxed{3x_3 = 3}$ (or $x_3 = 1$).

Knowing how to find this equation allows us to solve two problems involving lines in $\mathbb{R}^3$:

(1) If two lines intersect, they lie in a common plane. What's that plane's equation?

(2) If two lines don't intersect, how far apart are they?

(1) Plane Containing Intersecting Lines.

Consider the lines

$$① \quad \vec{x} = (1,3,2) + t(1,0,2) \qquad t \in \mathbb{R}$$

$$② \quad \vec{x} = (1,3,2) + s(-1,2,1) \qquad s \in \mathbb{R}$$

They intersect, since both contain $(1,3,2)$.
So, they both lie in some plane. What's this plane's normal vector?

It should be perpendicular to both direction vectors,
so $\vec{n} = (1,0,2) \times (-1,2,1)$.

Exercise: Show that $\vec{n} = (-4, -3, 2)$.

The plane: $-4x_1 - 3x_2 + 2x_3 = d$.

To get d, plug in $(1, 3, 2)$.

$$-4(1) - 3(3) + 2(2) = d \quad \Rightarrow \quad d = -9.$$

So... $\boxed{-4x_1 - 3x_2 + 2x_3 = -9}$ is the plane containing both lines.

(2) Distance Between Non-Intersecting Lines

Now consider the lines

$$① \quad \vec{x} = (2, -3, 1) + t(1, 1, 3), \qquad t \in \mathbb{R}$$

$$② \quad \vec{x} = (1, 4, 2) + s(2, 0, 1), \qquad s \in \mathbb{R}$$

These lines do not intersect! Exercise: check!
So what's the distance between them?

Idea: Find a plane containing the first line that is parallel to the second line.

The normal vector is the cross product of the direction vectors:

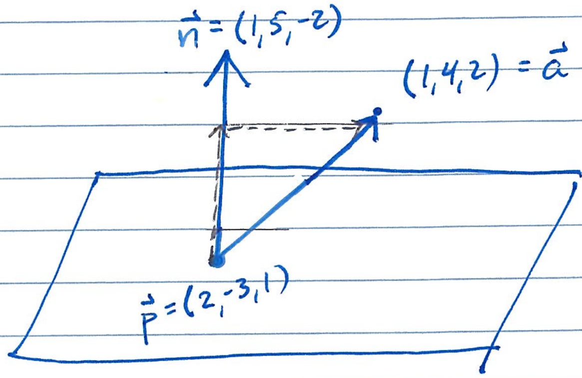
$$\vec{n} = (1, 1, 3) \times (2, 0, 1) = (1, 5, -2) \qquad \text{(check!)}$$

Since $(2, -3, 1)$ is on the plane, we get

$$x_1 + 5x_2 - 2x_3 = d \quad \Rightarrow \quad (2) + 5(-3) - 2(1) = d$$
$$\Rightarrow \quad d = -15$$

The equation of the plane is $x_1 + 5x_2 - 2x_3 = -15$

Now we'll find the distance between this plane and <u>any point</u> on the second line, say $(1,4,2)$.



(see §1.3)

$$\vec{pa} = (1,4,2) - (2,-3,1) = (-1,7,1)$$

$$\text{proj}_{\vec{n}}(\vec{pa}) = \left(\frac{(-1,7,1)\cdot(1,5,-2)}{(1,5,-2)\cdot(1,5,-2)}\right)(1,5,-2)$$

$$= \left(\frac{32}{30}\right)(1,5,-2)$$

The distance is $\|\text{proj}_{\vec{n}}(\vec{pa})\| = \frac{32}{30}\|(1,5,-2)\|$

$$= \frac{32}{30}\sqrt{30} = \boxed{\frac{32}{\sqrt{30}}}$$

<u>WHEW!!</u>

<u>Final Topic:</u> Find the line of intersection of two planes.

Two intersecting planes meet in a line (not just at a point!) How can we find that line's equation?

Ex: Find the equation of the line formed by the intersection of the planes

$$x_1 + x_2 - 2x_3 = 3, \quad \text{and} \quad 2x_1 - x_2 + 3x_3 = 6$$

Solution:

What's the direction vector, $\vec{d}$? Well, it is parallel to both planes, and hence orthogonal to both normal vectors.

Thus, $\vec{d}$ is the cross product of the normal vectors!

$$\vec{d} = (1,1,-2) \times (2,-1,3) = \cdots = (1,-7,-3)$$

The equation of the line is

$$\vec{x} = \vec{p} + t(1,-7,-3), \quad t \in \mathbb{R}$$

where $\vec{p}$ is any point on the line (i.e., any point in both our planes).

$$\boxed{\text{To get } \vec{p}, \text{ set } x_3 = 0 \text{ and use the plane equations to get } x_1 \text{ and } x_2.}$$

When $x_3 = 0$, $x_1 + x_2 - 2x_3 = 3 \Rightarrow x_1 + x_2 = 3$ (1)

$2x_1 - x_2 + 3x_3 = 6 \Rightarrow 2x_1 - x_2 = 6$ (2)

From (1), $x_1 = 3 - x_2$.

From (2), $2x_1 - x_2 = 6 \Rightarrow 2(3 - x_2) - x_2 = 6$

$\Rightarrow \cancel{6} - 3x_2 = \cancel{6}$

$\Rightarrow x_2 = 0$

So $x_1 = 3 - x_2 = 3 - 0 = 3$. Thus, $\vec{p} = (3,0,0)$

$$\boxed{\text{Line: } (3,0,0) + t(1,-7,-3), \quad t \in \mathbb{R}.}$$