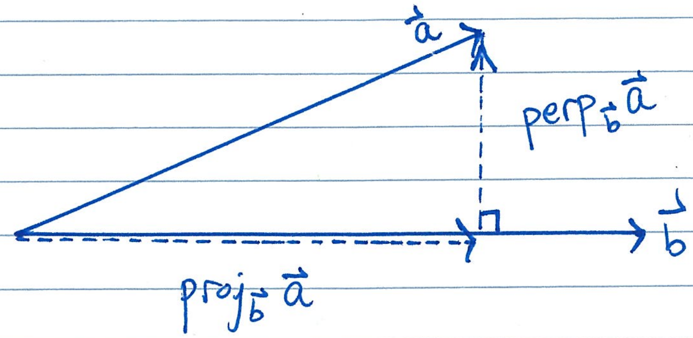
§1.3 - Projections and Minimum Distance.

Suppose we have 2 vectors, $\vec{a}$ and $\vec{b}$. We can write $\vec{a}$ as a sum of two special vectors:

1. one in the same direction as $\vec{b}$, and
2. one perpendicular to $\vec{b}$.



The first vector is called the <u>projection of $\vec{a}$ onto $\vec{b}$</u>, and is denoted $\text{proj}_{\vec{b}}\vec{a}$.

The second vector is called the <u>perpendicular part</u>, and is denoted $\text{perp}_{\vec{b}}\vec{a}$.

<u>Formula:</u>

$$\boxed{\text{proj}_{\vec{b}}\vec{a} = \underbrace{\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)}\vec{b} = \underbrace{\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)}\vec{b}}$$

these are real numbers

Since $\vec{a} = \text{proj}_{\vec{b}}\vec{a} + \text{perp}_{\vec{b}}\vec{a}$, we get

$$\boxed{\text{perp}_{\vec{b}}\vec{a} = \vec{a} - \text{proj}_{\vec{b}}\vec{a}}$$

Ex: Let's find the projection of $(3,1,6)$ onto $(1,2,1)$!

$$\text{Proj}_{(1,2,1)}(3,1,6) = \left[\frac{(3,1,6)\cdot(1,2,1)}{(1,2,1)\cdot(1,2,1)}\right](1,2,1)$$

$$= \left[\frac{3+2+6}{1+4+1}\right](1,2,1)$$

$$= \frac{11}{6}(1,2,1) = \boxed{\left(\frac{11}{6}, \frac{11}{3}, \frac{11}{6}\right)}$$

Ex: Find $\text{perp}_{(1,3)}(1,1)$.

Solution: $\text{proj}_{(1,3)}(1,1) = \left[\frac{(1,1)\cdot(1,3)}{(1,3)\cdot(1,3)}\right](1,3)$

$$= \left[\frac{1+3}{1+9}\right](1,3)$$

$$= \frac{4}{10}(1,3) = \left(\frac{2}{5}, \frac{6}{5}\right)$$

So... $\text{perp}_{(1,3)}(1,1) = (1,1) - \text{proj}_{(1,3)}(1,1)$

$$= (1,1) - \left(\frac{2}{5}, \frac{6}{5}\right)$$

$$= \boxed{\left(\frac{3}{5}, -\frac{1}{5}\right)}$$

Properties of Proj and Perp:

1. $\text{Proj}_{\vec b}(\vec a_1 + \vec a_2) = \text{Proj}_{\vec b}\,\vec a_1 + \text{Proj}_{\vec b}\,\vec a_2$

2. $\text{Proj}_{\vec b}(k\vec a) = k\,\text{Proj}_{\vec b}\,\vec a \qquad (k \in \mathbb{R})$

3. $\text{Proj}_{\vec b}\left(\text{Proj}_{\vec b}\,\vec a\right) = \text{Proj}_{\vec b}(\vec a)$

   (i.e., we get nothing new by projecting again!)

4. $\text{Proj}_{\vec b}\,\vec a \cdot \text{Perp}_{\vec b}\,\vec a = 0$

   (i.e., perp and proj are indeed orthogonal!)

Proof of 4:

$$\text{Proj}_{\vec b}\,\vec a \cdot \text{Perp}_{\vec b}\,\vec a = \left(\left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)\vec b\right) \cdot \left(\vec a - \left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)\vec b\right)$$

$$= \left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)\left[\vec b \cdot \left(\vec a - \underbrace{\left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)}_{\text{real \#}}\vec b\right)\right]$$

$$= \left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)\left[\vec b\cdot\vec a - \left(\frac{\vec a\cdot\vec b}{\cancel{\vec b\cdot\vec b}}\right)\cancel{(\vec b\cdot\vec b)}\right]$$

$$= \left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)\left(\vec b\cdot\vec a - \vec a\cdot\vec b\right)$$

$$= \left(\frac{\vec a\cdot\vec b}{\vec b\cdot\vec b}\right)(\vec 0) = 0, \text{ as claimed.}$$

$\blacksquare$

Exercise: Prove properties 1., 2., 3.

Okay... so what can we do with proj and perp?

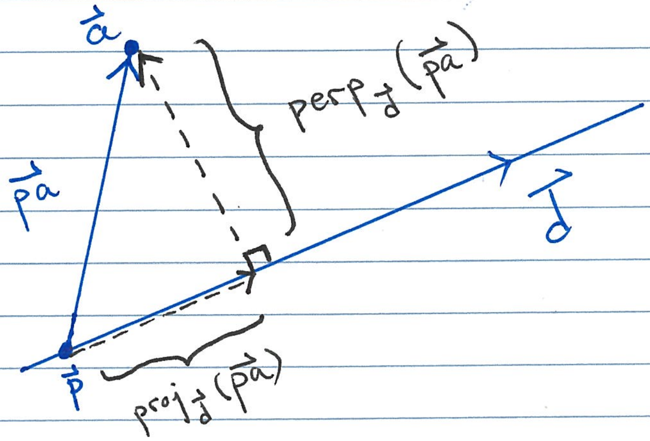
These vectors can help us find

1. the shortest distance (and closest point) from a point to a line

2. the shortest distance (and closest point) from a point to a plane.

Distance from Point to Line

Suppose we have a point $\vec{a}$ and a line $\vec{x} = \vec{p} + t\vec{d}$. What is the distance from $\vec{a}$ to this line?



It looks like the distance is $\|\text{perp}_{\vec{d}}(\vec{pa})\|$.

The closest point is $\vec{p} + \text{proj}_{\vec{d}}(\vec{pa})$

(or we can use $\vec{a} - \text{perp}_{\vec{d}}(\vec{pa})$)

Ex: Find the shortest distance from $\vec{a} = (0,0,2)$ to the line $(2,1,0) + t(-1,2,2)$, $t \in \mathbb{R}$.

Solution: Here
$$\vec{a} = (0,0,2)$$
$$\vec{p} = (2,1,0)$$
$$\vec{d} = (-1,2,2)$$

So $\vec{pa} = \vec{a} - \vec{p} = (0,0,2) - (2,1,0) = (-2,-1,2)$.

We have

$$\text{Proj}_{\vec{d}}(\vec{pa}) = \left(\frac{(-2,-1,2)\cdot(-1,2,2)}{(-1,2,2)\cdot(-1,2,2)}\right)(-1,2,2)$$

$$= \left(\frac{2-2+4}{1+4+4}\right)(-1,2,2)$$

$$= \frac{4}{9}(-1,2,2) = \left(\frac{-4}{9}, \frac{8}{9}, \frac{8}{9}\right)$$

So $\text{Perp}_{\vec{d}}(\vec{pa}) = \vec{pa} - \text{proj}_{\vec{d}}(\vec{pa})$

$$= (-2,-1,2) - \left(\frac{-4}{9}, \frac{8}{9}, \frac{8}{9}\right)$$

$$= \left(\frac{-14}{9}, \frac{-17}{9}, \frac{10}{9}\right)$$

The distance is $\|\text{perp}_{\vec{d}}(\vec{pa})\| = \sqrt{\left(\frac{14}{9}\right)^2 + \left(\frac{-17}{9}\right)^2 + \left(\frac{10}{9}\right)^2}$

$$= \boxed{\frac{\sqrt{65}}{3}}$$

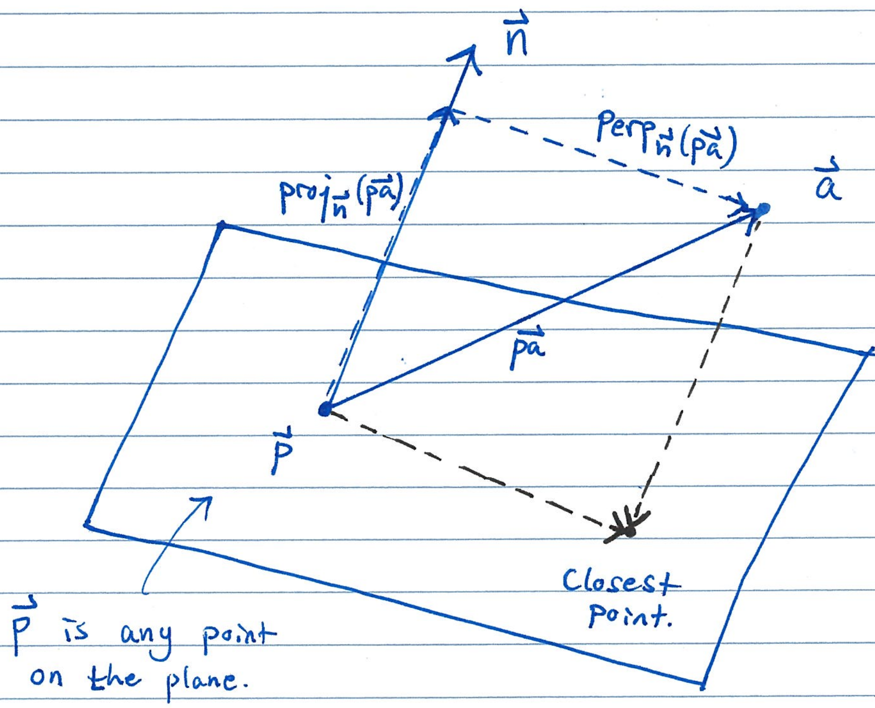
The closest point is

$$\vec{p} + \mathrm{proj}_{\vec{d}}(\vec{pa}) = (2,1,0) + \left(\frac{-4}{9}, \frac{8}{9}, \frac{8}{9}\right)$$

$$= \boxed{\left(\frac{14}{9}, \frac{17}{9}, \frac{8}{9}\right)}$$

Distance from Point to Plane

Suppose we have a point $\vec{a}$ and a plane with normal vector $\vec{n}$. What is the shortest distance from $\vec{a}$ to this plane?



It looks like the distance is $\|\mathrm{proj}_{\vec{n}}(\vec{pa})\|$.

The closest point is $\vec{a} - \mathrm{proj}_{\vec{n}}(\vec{pa})$

(we could also use $\vec{p} + \mathrm{perp}_{\vec{n}}(\vec{pa})$)

Ex: Find the shortest distance from $\vec{a} = (1,0,1)$ to the plane $x_1 + 2x_2 - 3x_3 = 1$. What is the closest point?

Solution: Our normal vector is $\vec{n} = (1,2,-3)$.

For $\vec{p}$, just take any point on the plane. Since $x_1 + 2x_2 - 3x_3 = 1$, we can take $\vec{p} = (1,0,0)$.

So $\vec{pa} = \vec{a} - \vec{p} = (1,0,1) - (1,0,0) = (0,0,1)$.

$$\Rightarrow \text{proj}_{\vec{n}}(\vec{pa}) = \left[\frac{(0,0,1)\cdot(1,2,-3)}{(1,2,-3)\cdot(1,2,-3)}\right](1,2,-3)$$

$$= \frac{-3}{14}(1,2,-3)$$

$$= \left(\frac{-3}{14}, \frac{-3}{7}, \frac{9}{14}\right)$$

Then the shortest distance is

$$\|\text{proj}_{\vec{n}}(\vec{pa})\| = \sqrt{\left(\frac{-3}{14}\right)^2 + \left(-\frac{3}{7}\right)^2 + \left(\frac{9}{14}\right)^2}$$

$$= \sqrt{\frac{126}{196}} = \boxed{\frac{3}{\sqrt{14}}}$$

The closest point is

$$\vec{a} - \text{proj}_{\vec{n}}(\vec{pa}) = (1,0,1) - \left(\frac{-3}{14}, \frac{-3}{7}, \frac{9}{14}\right)$$

$$= \boxed{\left(\frac{17}{14}, \frac{3}{7}, \frac{5}{14}\right)}$$

Ex: Find the shortest distance from $\vec{a} = (1,1,1)$ to the plane $2x_1 - x_2 + x_3 = 2$. What is the closest point?

Solution: $\vec{n} = (2,-1,1)$, and $\vec{p} = (1,0,0)$ is a point on our plane.

$$\vec{pa} = \vec{a} - \vec{p} = (1,1,1) - (1,0,0) = (0,1,1).$$

So
$$\text{Proj}_{\vec{n}}(\vec{pa}) = \left[\frac{(0,1,1)\cdot(2,-1,1)}{(2,-1,1)\cdot(2,-1,1)}\right](2,-1,1)$$

$$= \left(\frac{0}{6}\right)(2,-1,1) = (0,0,0)$$

(Weird, but let's roll with it)

The distance is $\|\text{proj}_{\vec{n}}(\vec{pa})\| = \|\vec{0}\| = 0$... hmm.

If the distance is 0, $\vec{a}$ must have already been on the plane!

Indeed, if we plug $\vec{a} = (1,1,1)$ into $2x_1 - x_2 + x_3 = 2$, we get

$$2(1) - (1) + (1) = 2 \quad \checkmark$$

So the distance is 0 and the closest point is $\vec{a}$!