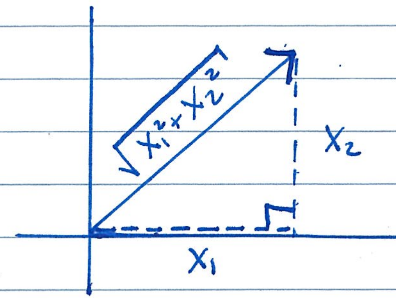
§1.2 - Length, Dot Product, Planes

The length of a vector

$\vec{x} = (x_1, x_2)$ in $\mathbb{R}^2$



is defined as

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$$

For $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$, the length is

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

and in general, $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

The distance between $\vec{a}$ and $\vec{b}$ is

$$\|\overrightarrow{ab}\| = \|\vec{b} - \vec{a}\|$$

Ex: The distance from $\vec{a} = (1,1)$ to $\vec{b} = (2,4)$ is

$$\|(2,4) - (1,1)\| = \|(1,3)\|$$

$$= \sqrt{1^2 + 3^2} = \boxed{\sqrt{10}}$$

Properties of $\|\cdot\|$:

1. $\|\vec{x}\| > 0$, unless $\vec{x} = 0$; $\|\vec{0}\| = 0$.
2. $\|k\vec{x}\| = |k| \cdot \|\vec{x}\|$ for $k \in \mathbb{R}$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for any $\vec{x}, \vec{y}$ (triangle inequality)

Another operation is the <u>dot product</u>:

$$(a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2$$

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vdots$$

$$(a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

<u>Ex</u>: $(1, -1, 2) \cdot (3, 4, 5) = (1)(3) + (-1)(4) + (2)(5)$

$= 3 - 4 + 10 = \boxed{9}$

<u>Properties of the Dot Product</u>:

1. $\boxed{\vec{a} \cdot \vec{a} = \|a\|^2} = a_1^2 + a_2^2 + \cdots + a_n^2$

   This will be used a lot!

2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$ for $k \in \mathbb{R}$.

5. $\boxed{\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta}$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

<u>Note</u>: $\vec{a}$ and $\vec{b}$ are <u>perpendicular</u> (<u>orthogonal</u>) if $\theta = \pi/2$

(i.e., if they meet at a $90°$ angle)

By property 5. of the dot product,

$$\boxed{\text{Non-zero } \vec{a} \text{ and } \vec{b} \text{ are orthogonal } \Leftrightarrow \vec{a}\cdot\vec{b} = 0.}$$

(Try to prove it!)

__Ex:__ Find the angle between $\vec{a} = (1,1,0)$ and $\vec{b} = (0,1,1)$.

__Solution:__ $\vec{a}\cdot\vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$, so...

$$(1,1,0)\cdot(0,1,1) = \|(1,1,0)\| \, \|(0,1,1)\| \cos\theta$$

$$\Rightarrow \cancel{(1)(0)} + (1)(1) + \cancel{(0)(1)} = \sqrt{1^2+1^2+\cancel{0^2}}\,\sqrt{\cancel{0^2}+1^2+1^2}\,\cos\theta$$

$$\Rightarrow 1 = \sqrt{2}\cdot\sqrt{2}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{2}, \quad \text{so } \boxed{\theta = \frac{\pi}{3}}$$

__Ex:__ What value(s) of k make $(1,k,3k)$ orthogonal to $(1,2,0)$?

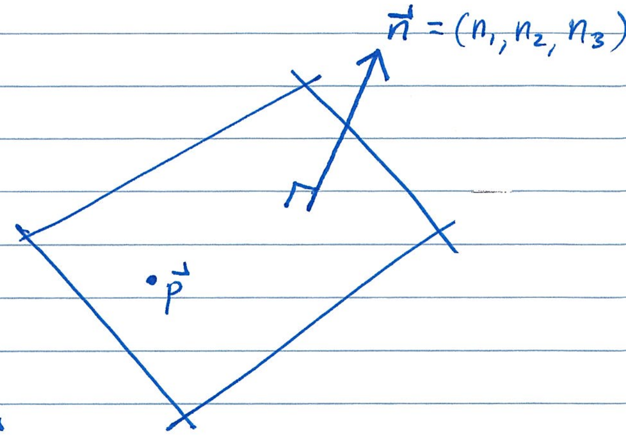
__Solution:__ They are orthogonal exactly when the dot product is 0.

$$(1,2,0)\cdot(1,k,3k) = (1)(1) + (2)(k) + \cancel{(0)(3k)}$$
$$= 1 + 2k$$

So $1 + 2k = 0 \Rightarrow 2k = -1$

$$\Rightarrow \boxed{k = -\frac{1}{2}}$$

Planes in $\mathbb{R}^3$



To define a plane in $\mathbb{R}^3$, we just need to specify

1. its <u>normal vector</u> $\vec{n}$
   (a vector perpendicular to the plane)

2. a point on the plane.

The <u>scalar equation of a plane</u> is

$$\boxed{n_1 x_1 + n_2 x_2 + n_3 x_3 = d}$$

where $\vec{n}$ is the normal vector and d is found by substituting (x_1, x_2, x_3) with a point on the plane.

<u>Ex:</u> Find the scalar equation of the plane with normal vector $(1, 4, -2)$, and containing $(2, 1, 2)$.

<u>Solution:</u> $\vec{n} = (1, 4, -2)$, so the equation is

$$1x_1 + 4x_2 - 2x_3 = d.$$

Plug in $(x_1, x_2, x_3) = (2, 1, 2)$ to get d:

$$1(2) + 4(1) - 2(2) = d \Rightarrow d = 2.$$

The equation is $\boxed{x_1 + 4x_2 - 2x_3 = 2}$

Two planes are parallel if the normal vector of one plane is a non-zero scalar multiple of the normal vector of the other.

Ex:

$$2x_1 - x_2 + 3x_3 = 5 \quad \text{and} \quad 4x_1 - 2x_2 + 6x_3 = 1$$

are parallel planes. Why? Their normal vectors are $(2,-1,3)$ and $(4,-2,6)$, and

$$(4,-2,6) = 2(2,-1,3).$$

(the values of d don't matter!)

Two planes are orthogonal (perpendicular) if their normal vectors are orthogonal (dot product $= 0$).

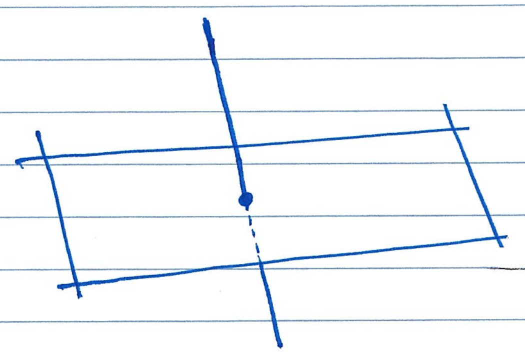
Ex:

Are the planes $x_1 + 3x_2 - 4x_3 = 8$ and $-x_1 + 3x_2 + 2x_3 = 0$ orthogonal?

Normal vectors: $(1,3,-4)$, $(-1,3,2)$

$(1,3,-4)\cdot(-1,3,2) = -1 + 9 - 8 = 0.$ Yes!

Q: Given a line and a plane in $\mathbb{R}^3$, how do we determine the point at which they intersect?



A: Find the parametric equation of the line. Plug these into the plane and solve for t.

Ex: Find the intersection of the plane $x_1 + x_2 + 2x_3 = 4$ and the line $(1,2,1) + t(1,2,-1)$, $t \in \mathbb{R}$.

Solution: The parametric equation is

$$x_1 = 1+t, \quad x_2 = 2+2t, \quad x_3 = 1-t$$

So $x_1 + x_2 + 2x_3 = 4 \Rightarrow (1+t) + (2+2t) + 2(1-t) = 4$

$$\Rightarrow 5 + t = 4$$

$$\Rightarrow t = -1$$

So $x_1 = 1+t = 0$
$x_2 = 2+2t = 0$
$x_3 = 1-t = 2.$

They intersect at $\boxed{(0,0,2).}$