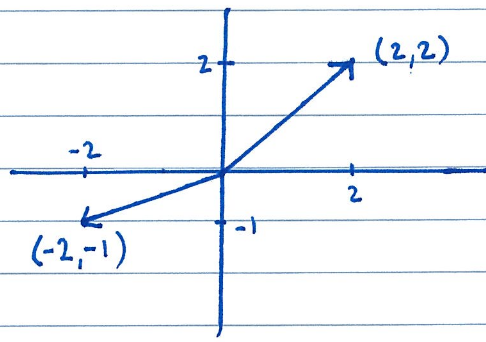
Chapter 1 – Vector Algebra

§1.1 – Vectors in 2- and 3-Dimensions

A vector in $\mathbb{R}^2$ (2-dimensional space) is an ordered pair

$$\vec{x} = (x_1, x_2)$$



Hey Zack ... aren't these just like points ??
Yes! But sometimes it's helpful to think of them as arrows starting at the origin!

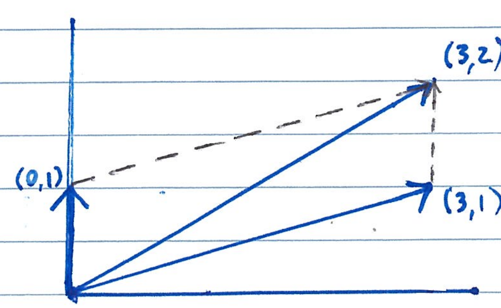
Note: The zero vector is $\vec{0} = (0,0)$

What can we do with vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$?

1. Vector Addition

$$\vec{a} + \vec{b} = (a_1 + a_2, b_1 + b_2)$$

Ex: $(0,1) + (3,1) = (3,2)$

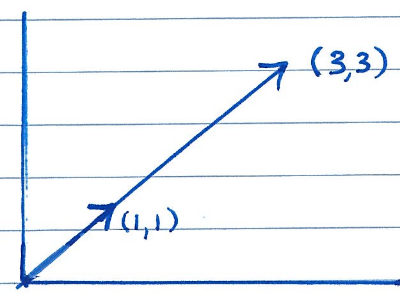


[Put one vector after the other to add visually!]

2. Scalar Multiplication (i.e., multiply a vector by a real number.)

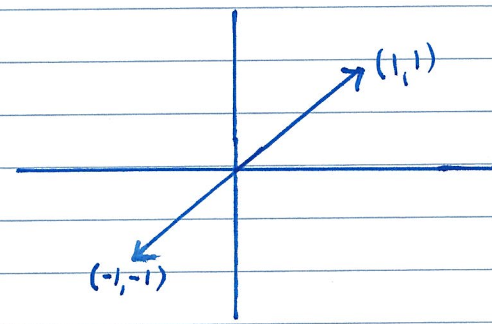
For $c \in \mathbb{R}$, $c\vec{a} = c(a_1, a_2) = (ca_1, ca_2)$.

Ex: $3(1,1) = (3,3)$



The vector was stretched by a factor of 3!

Ex: $-(1,1) = (-1,-1)$



A negative number reverses the vector's direction!

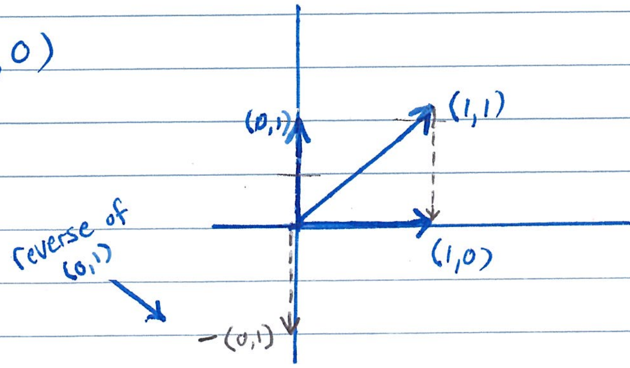
3. Vector Subtraction

$$\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2),$$

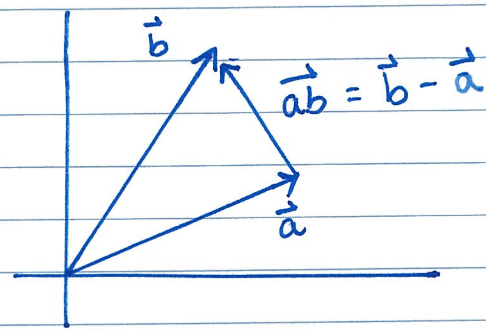
This is really just a combination of addition, and multiplication by -1.

Visually, reverse $\vec{b}$, then add to $\vec{a}$.

Ex: $(1,1) - (0,1) = (1,0)$



Note: The vector starting at $\vec{a}$ and ending at $\vec{b}$ is $\vec{b} - \vec{a}$. Sometimes we denote this as $\vec{ab}$.

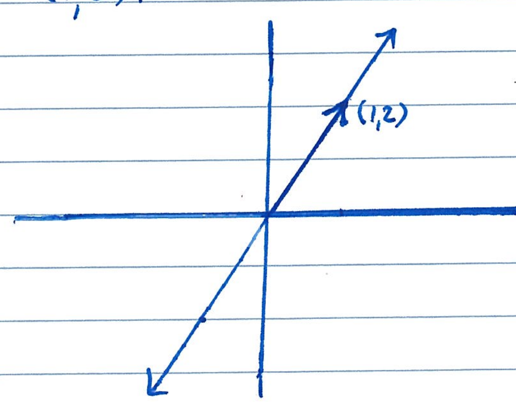


Lines in $\mathbb{R}^2$

The set of vectors of the form

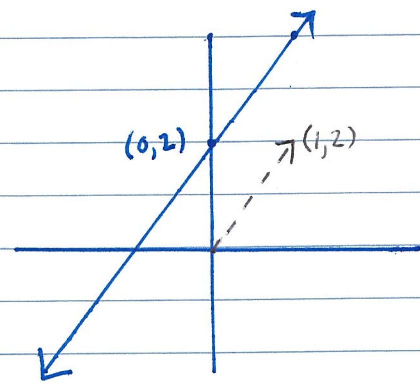
$$t(1,2), \quad t \in \mathbb{R}$$

describes a line through the origin, and in the same direction as $(1,2)$.



What about $(0,2) + t(1,2), \quad t \in \mathbb{R}$?

This new line passes through $(0,2)$, but moves in the same direction as (i.e., parallel to) $(1,2)$.



In general, the line passing through $\vec{p}$ and in the same direction as $\vec{d}$ is

$$\boxed{\vec{x} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R}}$$

Point on the line — Direction vector

This is called the <u>vector equation of the line</u>.

Alternatively, we could have described each coordinate separately:

If $\vec{p} = (p_1, p_2)$ and $\vec{d} = (d_1, d_2)$, then

$$\vec{x} = \vec{p} + t\vec{d} \Rightarrow (x_1, x_2) = (p_1, p_2) + t(d_1, d_2)$$

$$\Rightarrow \begin{cases} x_1 = p_1 + t d_1 \\ x_2 = p_2 + t d_2 \end{cases}$$

This is called the <u>parametric equation of the line</u>

<u>Ex</u>: Find the vector and parametric equations of

(a) the line through $(1,1)$ and $(2,4)$.

Solution:

What's the direction vector?

It's the vector starting at $\vec{a} = (1,1)$ and ending at $\vec{b} = (2,4)$. That is

$$\vec{d} = \vec{ab} = \vec{b} - \vec{a} = (2,4) - (1,1) = (1,3)$$

(We could have also used $(1,1) - (2,4)$.)

Pick either $(1,1)$ or $(2,4)$ to be $\vec{p}$.

So... $\boxed{\vec{x} = (1,1) + t(1,3), \quad t \in \mathbb{R}}$

is the vector equation.

Parametric: $\boxed{\begin{cases} x_1 = 1 + t \\ x_2 = 1 + 3t \end{cases} \quad t \in \mathbb{R}}$

(b). the line through $(3,1)$ and parallel to $\vec{x} = (-1,2) + t(4,-1)$, $t \in \mathbb{R}$.

Solution: Parallel means same direction! So, $\vec{d} = (4,-1)$.

Vector equation: $\boxed{\vec{x} = (3,1) + t(4,-1), \quad t \in \mathbb{R}}$

Parametric equation: $\boxed{\begin{cases} x_1 = 3 + 4t \\ x_2 = 1 - t \end{cases}, \quad t \in \mathbb{R}.}$

Note: We can do the exact same stuff in higher dimensions!

$\mathbb{R}^3$ = set of all triples (x_1, x_2, x_3)

$\mathbb{R}^4$ = set of all quadruples (x_1, x_2, x_3, x_4) etc.

Ex: Find the vector and parametric equations of the line through $(1, 0, 3)$ with direction vector $(0, 3, -5)$.

Solution: $\vec{d} = (0,3,-5) - (1,0,3) = (-1,3,-8)$

Vector equation: $\boxed{\vec{x} = (1,0,3) + t(-1,3,-8), \quad t \in \mathbb{R}.}$

Parametric equation: $\boxed{\begin{cases} x_1 = 1 - t \\ x_2 = 3t \\ x_3 = 3 - 8t \end{cases} \quad t \in \mathbb{R}}$

Intersection of Lines

Suppose we have 2 lines

① $(5,3) + t(1,-2), \quad t \in \mathbb{R}$
② $(2,1) + s(1,2), \quad s \in \mathbb{R}$

Where do they intersect?

Equate their parametric equations and solve for t and s!

Parametric Equations:

① $\begin{cases} x_1 = 5 + t \\ x_2 = 3 - 2t \end{cases}$ ② $\begin{cases} x_1 = 2 + s \\ x_2 = 1 + 2s \end{cases}$

So $5 + t = 2 + s$
$3 - 2t = 1 + 2s$

From the first equation, we get

$$5 + t = 2 + s \Rightarrow 3 + t = s$$

Replace the s in the second equation and solve:

$$\begin{aligned} 3 - 2t = 1 + 2s \quad &\Rightarrow 3 - 2t = 1 + 2(3 + t) \\ &\Rightarrow 3 - 2t = 1 + 6 + 2t \\ &\Rightarrow 3 - 2t = 7 + 2t \\ &\Rightarrow -4 = 4t \\ &\Rightarrow t = -1 \end{aligned}$$

Okay! So the lines cross when $t = -1$.

But $x_1 = 5 + t \Rightarrow x_1 = 5 + (-1) = 4$
$x_2 = 3 - 2t \Rightarrow x_2 = 3 - 2(-1) = 5$

$\boxed{\text{The lines meet at } (4,5).}$

<u>Note</u>: It's possible that the lines don't meet at all.

In this case we get $0 = 1$ (or something else equally awful) when solving for s or t.