§8.3 – Trigonometric Integrals

We'll begin by looking at integrals of the form

$$\boxed{\int \sin^m x \cos^n x \, dx}$$

where m and n are non-negative integers.

Warm up: $\int \sin^m x \cos x \, dx$

Sub $u = \sin x$, $du = \cos x \, dx$ to get $\int u^m \, du$

(Easy!)

What about $\int \sin^m x \cos^3 x \, dx$?

Again, sub $u = \sin x$, $du = \cos x \, dx$ to get

$$\int u^m \cos^2 x \cdot \cos x \, dx = \int u^m (1 - \sin^2 x) \, du$$

$$= \int u^m (1 - u^2) \, du \quad \text{(Easy!)}$$

This strategy will work whenever n is odd! If instead m is odd, we'll set $u = \cos x$.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(i) If m is odd, let $u = \cos x$.

(ii) If n is odd, let $u = \sin x$.

If both are odd, let u be the function with the higher power!

(iii) If m & n are even, use the identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x), \quad \cos^2 x = \tfrac{1}{2}(1+\cos 2x).$$

Ex: (a) $\int \sin^3 x \cdot \cos^8 x \, dx$

Solution: m odd $\Rightarrow$ $u = \cos x$, $du = -\sin x \, dx$, giving

$$\int \sin^2 x \cdot u^8 \cdot \underbrace{\sin x \, dx}_{=-du} = -\int (1-\cos^2 x)\cdot u^8 \, du$$

$$= -\int (1-u^2)\, u^8 \, du$$

$$= \int (u^{10} - u^{8})\, du$$

$$= \frac{u^{11}}{11} - \frac{u^{9}}{9} + C$$

$$= \boxed{\frac{\cos^{11} x}{11} - \frac{\cos^{9} x}{9} + C}$$

(b) $\int \sin^{15} x \cdot \cos^{5} x \, dx$

Solution: Both powers are odd, but $\sin x$ has the bigger power. Let $u = \sin x$, $du = \cos x \, dx$, giving

$$\int u^{15} \cos^{4} x \cdot \underbrace{\cos x \, dx}_{= du} = \int u^{15} (1-u^2)^2 \, du$$

If instead we used $u = \cos x$, we would need to expand $(1-u^2)^7$... Ew.

$$= \int u^{15} (1 - 2u^2 + u^4)\, du$$

$$= \int (u^{15} - 2u^{17} + u^{19})\, du$$

$$= \frac{u^{16}}{16} - \frac{2u^{18}}{18} + \frac{u^{20}}{20} + C$$

$$= \boxed{\frac{\sin^{16} x}{16} - \frac{\sin^{18} x}{9} + \frac{\sin^{20} x}{20} + C}$$

(c) $\displaystyle\int \sin^2 x \cos^2 x \, dx$

Solution: Both powers are even, so use the double angle identities.

$$\int \sin^2 x \cos^2 x \, dx = \int \tfrac{1}{2}(1-\cos 2x) \cdot \tfrac{1}{2}(1+\cos 2x) \, dx$$

$$= \frac{1}{4} \int \left(1 - \cos^2(2x)\right) dx$$

$$= \frac{1}{4} \int \left(1 - \tfrac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4} \int \left(\frac{1}{2} - \frac{\cos 4x}{2}\right) dx$$

$$= \frac{1}{8} \int (1 - \cos 4x) \, dx$$

Could do a u-sub ($u = 4x$) or note that $\int \cos(nx)dx = \frac{\sin(nx)}{n} + C$!

$$= \boxed{\frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C}$$

Let's now look at integrals of the form

$$\boxed{\int \tan^m x \cdot \sec^n x \, dx}$$

where again, m and n are non-negative integers.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(i) If n is even, let $u = \tan x$ and use $\sec^2 x = \tan^2 x + 1$.

(ii) If m is odd, let $u = \sec x$ and use $\tan^2 x = \sec^2 x - 1$.

If n is even and m is odd, let u be the function with the higher power!

Ex: $\int \tan^3 x \sec^7 x \, dx$

Solution: m is odd, so let $u = \sec x$. We have

$du = \sec x \tan x \, dx$, hence

$$\int \tan^3 x \sec^7 x \, dx = \int \tan^2 x \sec^6 x \cdot \underbrace{\sec x \tan x \, dx}_{du}$$

$$= \int (\sec^2 x - 1)\, u^6 \, du$$

$$= \int (u^2 - 1)\, u^6 \, du$$

$$= \int (u^8 - u^6)\, du$$

$$= \frac{u^9}{9} - \frac{u^7}{7} + C = \boxed{\frac{\sec^9 x}{9} - \frac{\sec^7 x}{7} + C}$$

Ex: $\displaystyle\int \tan^4 x \sec^4 x \, dx$

Solution: n is even, so let $u = \tan x$, $du = \sec^2 x \, dx$.

We have

$$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \cdot \underbrace{\sec^2 x \, dx}_{du}$$

$$= \int u^4 (\tan^2 x + 1)\, du$$

$$= \int u^4 (u^2 + 1)\, du$$

$$= \int (u^6 + u^4)\, du$$

$$= \frac{u^7}{7} + \frac{u^5}{5} + C = \boxed{\frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C}$$

What about cases like

$$\int \tan^2 x \sec x \, dx$$

where m is even and n is odd?

Step 1: Use $\tan^2 x = \sec^2 x - 1$ to get only powers of $\sec x$:

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \cdot \sec x \, dx = \int (\sec^3 x - \sec x) \, dx$$

Step 2: Integrate powers of $\sec x$ as follows:

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C} \qquad \boxed{\int \sec^2 x \, dx = \tan x + C}$$

and for integers $n \geq 3$,

$$\boxed{\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx}$$

[This is called a "reduction formula" for powers of secant. See end of notes for a proof!]

Example using the Reduction Formula:

$n=4$

$$\int \sec^4 x \, dx = \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \int \sec^2 x \, dx$$

$$= \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x + C$$

Okay, back to our original problem!

Ex: $\int \tan^2 x \sec x \, dx$

Solution: $\int \tan^2 x \sec x \, dx = \int (\sec^3 x - \sec x) \, dx$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \left[\frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx \right] - \int \sec x \, dx$$

Reduction formula!

$$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \int \sec x \, dx$$

$$= \boxed{\frac{1}{2} \sec x \tan x - \frac{1}{2} \ln \left| \sec x + \tan x \right| + C}$$

Appendix: Proof of the Reduction Formula

To calculate $\int \sec^n x \, dx$, use IBP with

$u = \sec^{n-2} x$	$v = \tan x$
$du = (n-2) \sec^{n-3} x \cdot \sec x \tan x \, dx$ $= (n-2) \sec^{n-2} x \tan x \, dx$	$dv = \sec^2 x \, dx$

$$\int \sec^n x\, dx = \sec^{n-2}x \tan x - \int (n-2)\sec^{n-2}x \tan^2 x\, dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x\,(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2}x \tan x \underbrace{- (n-2)\int \sec^n x\, dx}_{\text{This is a multiple of the original integral! Move to LHS!}} + (n-2)\int \sec^{n-2}x\, dx$$

$$\Rightarrow (n-1)\int \sec^n x\, dx = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2}x\, dx$$

$\div (n-1)$

$$\Longrightarrow \int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx \qquad \blacksquare$$

"End of Proof"