Taylor and Maclaurin Series

Recall that Taylor's inequality states that the error in approximating $f(x)$ with $T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ satisfies

$$|\text{Error}| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

The $(n+1)!$ in the denominator suggests that the error may shink as the degree of $T_{n,a}(x)$ increases, which begs the question:

> What if we use infinitely many terms in our approximation?

We no longer have a polynomial ... we have a series!

Definition: The Taylor series for $f(x)$ centred at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

and if $a=0$, we call $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ the Maclaurin series of $f(x)$.

Remark: A Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is just a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ where the c_n's are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

for some function $f(x)$.

Ex: What is the Maclaurin series for $f(x)=e^x$?

Solution: $f^{(n)}(x)=e^x$, so $f^{(n)}(0)=e^0=1$ for all n.

Thus, the Maclaurin series is

$$\boxed{\sum_{n=0}^{\infty} \frac{\overset{1}{f^{(n)}(0)}}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

[Note: Since a Taylor or Maclaurin series is just a type of power series, we can use the ratio test to see where it converges!]

In our lesson on power series, we used the ratio test to show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

Ex: What is the Maclaurin series for $f(x) = \cos x$? What is its interval of convergence?

Solution:

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \quad \text{(Repeat!)} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 1 \quad \text{(Repeat!)} \end{aligned}$$

The Maclaurin series is

The even integers.

$$\boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

Using the ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}\, x^{2(n+1)}}{(2(n+1))!}}{\dfrac{\cancel{(-1)^{n}}\, x^{2n}}{(2n)!}} \right|$$

$$= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \cdot \frac{|x|^{2n+2}}{|x|^{2n}}$$

$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 \quad \text{for all } x \in (-\infty, \infty).$$

Since $L < 1$ always, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ converges for all x in $I = (-\infty, \infty)$, the interval of convergence.

The Maclaurin series for $\sin x$ is similar but includes only odd powers:

The odd integers.

$$\boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

Exercise: Use the ratio test to show that the Maclaurin series for $\sin x$ converges for all $x \in \mathbb{R}$

Another important example: Find the Maclaurin series and interval of convergence for $f(x) = \frac{1}{1-x}$

Solution: We compute $f^{(n)}(0)$:

$$\left.\begin{aligned} f(x) &= \frac{1}{1-x} \\ f'(x) &= \frac{1}{(1-x)^2} \\ f''(x) &= \frac{2\cdot 1}{(1-x)^3} \\ f'''(x) &= \frac{3\cdot 2\cdot 1}{(1-x)^4} \end{aligned}\right\}$$

In general, $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

$\Rightarrow f^{(n)}(0) = n!$ for all n.

Thus, the Maclaurin series for $\frac{1}{1-x}$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \boxed{\sum_{n=0}^{\infty} x^n}$$

$$= 1 + x + x^2 + x^3 + \cdots$$

which we know (from geometric series) converges when $|x| < 1$, or equivalently, for $x \in (-1,1)$.

Amazing fact: The functions e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$ are <u>exactly equal</u> to their Maclaurin series for x in the interval of convergence!

We have eliminated any error in our approximation!

Summary:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad I = (-\infty, \infty),\ R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad I = (-\infty, \infty),\ R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad I = (-\infty, \infty),\ R = \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad I = (-1, 1),\ R = 1$$

Note: We can plug in $x \in I$ to both sides of the above equalities to get some interesting results...

Ex: Subbing $x=1$ into $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ gives

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots = e^1 = e}$$

Ex: Subbing $x = \pi$ into $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$ gives

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots = \sin \pi = 0}$$

We'll see more exciting applications soon!