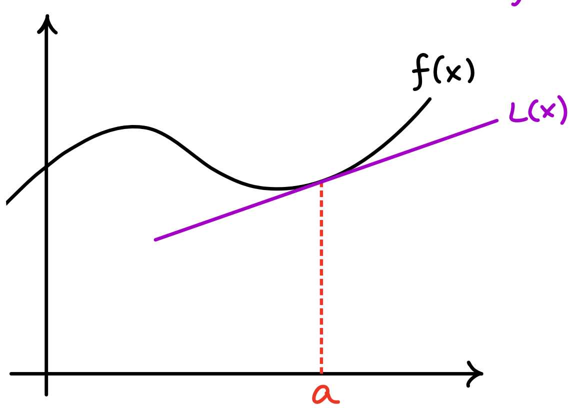
§ 10.3 – Taylor Polynomials (Series will be back soon!)

One of our main goals in MATH 118 is to be able to approximate complicated functions with simple functions: polynomials! We saw a case of this in MATH 116 when studying *linear approximations*.

Recall: The *linear approximation* to $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x-a)$$



← $L(x)$ is just the tangent line to $f(x)$ at $x = a$!

$$\boxed{L(x) \approx f(x) \text{ for } x \text{ near } a.}$$

Note:

$$\left.\begin{aligned} L(a) &= f(a) \\ L'(a) &= f'(a) \end{aligned}\right\}$$

L uses the values of $f(a)$ and $f'(a)$ to model $f(x)$!

Idea: Maybe we can obtain a better approximation by adding a squared term!

What should C be?

$$Q(x) = f(a) + f'(a)(x-a) + C(x-a)^2$$

Quadratic Approximation!

We can model the concavity of $f(x)$ by insisting that $Q''(a) = f''(a)$. This will allow us to determine C!

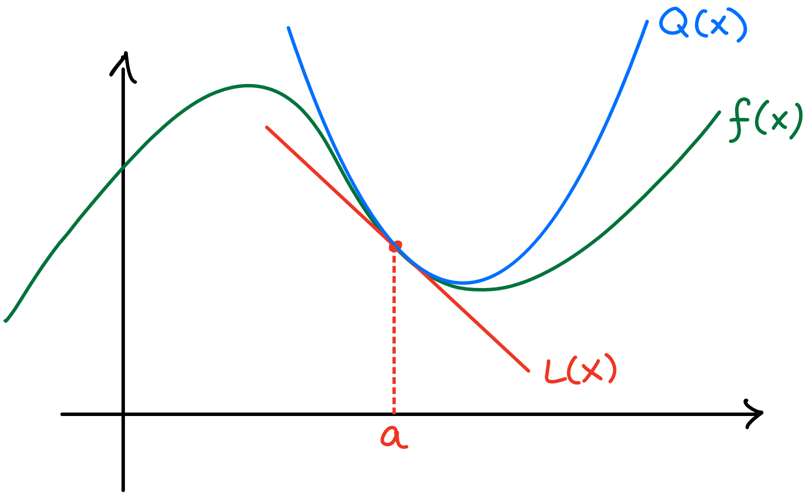
$$Q(x) = \underbrace{f(a)}_{\text{constant!}} + f'(a)(x-a) + C(x-a)^2$$

$$\Rightarrow Q'(x) = \underbrace{f'(a)}_{\text{constant!}} + 2C(x-a)$$

$$\Rightarrow Q''(x) = 2C$$

so $Q''(a) = 2C = f''(a) \Rightarrow C = \dfrac{f''(a)}{2}$.

$$\Rightarrow \boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$



Ex: Find the quadratic approximation to $f(x) = \sqrt{x}$ at $a = 1$.

Solution:

$f(x) = \sqrt{x} \quad \Rightarrow \quad f(1) = 1$

$f'(x) = \frac{1}{2} x^{-1/2} \quad \Rightarrow \quad f'(1) = \frac{1}{2}$

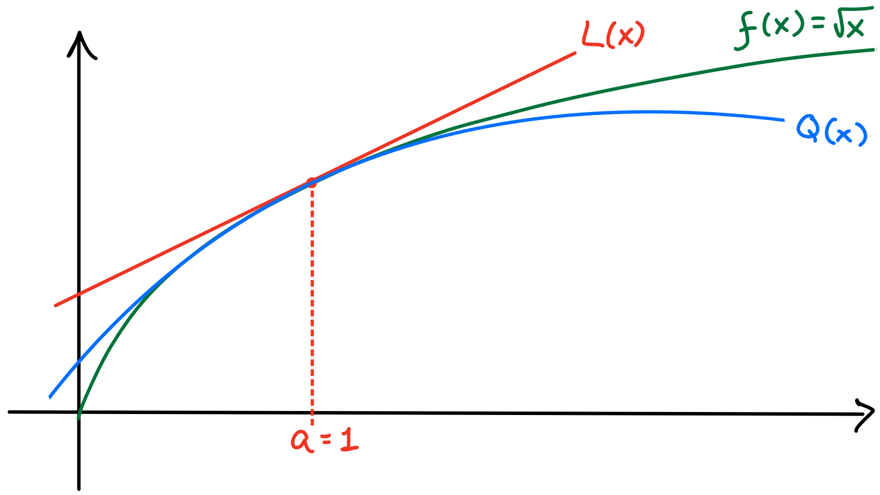
$f''(x) = \frac{-1}{4} x^{-3/2} \quad \Rightarrow \quad f''(1) = \frac{-1}{4}$

$$\Longrightarrow Q(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= \boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2}$$

For reference, the linear approximation is

$\underline{L(x) = 1 + \frac{1}{2}(x-1).}$



<u>But why stop here??</u>

We can approximate with a polynomial of degree 3 or 4 or 5 ... or really any degree n:

$$P(a) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

If we insist that $P^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \ldots, n$

(as we did for the quadratic approximation), we get

$$\boxed{c_k = \frac{f^{(k)}(a)}{k!}}$$

Thus, we get the following approximating polynomial.

Definition: The n^{th}-degree Taylor polynomial for $f(x)$ centred at $x=a$ is

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note: The linear approximation is $L(x) = T_{1,a}(x)$.

The quadratic approximation is $Q(x) = T_{2,a}(x)$.

We'll use the " $T_{n,a}(x)$ " notation from now on.

Ex: Find $T_{3,0}(x)$ for $f(x) = e^x$.

Solution: We need $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$ where $a=0$:

$$f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1$$

$$f'(x) = e^x \quad \Rightarrow \quad f'(0) = 1$$

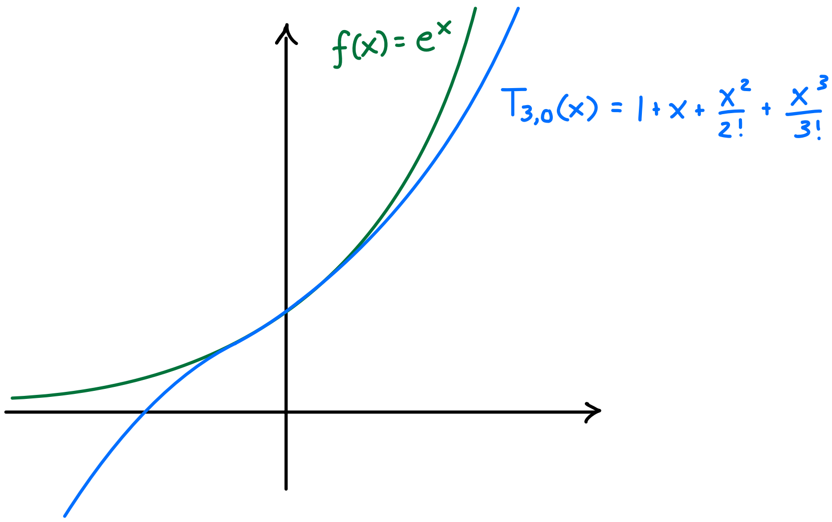
$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1$$

$$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = 1$$

We have

$$T_{3,0}(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$$



Remark: Taylor polynomials centred at $a = 0$ (i.e., $T_{n,0}(x)$) are often referred to as Maclaurin polynomials.

Ex: Find the 5^{th} - degree Maclaurin polynomial, $T_{5,0}(x)$ for $f(x) = \sin x$.

Solution:

$$f(x) = \sin x \Rightarrow f(0) = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1$$

$$\Rightarrow T_{5,0}(x) = \cancel{f(0)}^{=0} + \overbrace{f'(0)}^{=1}(x-0) + \frac{\cancel{f''(0)}^{=0}}{2!}(x-0)^2 + \frac{\overbrace{f'''(0)}^{=-1}}{3!}(x-0)^3 + \frac{\cancel{f^{(4)}(0)}^{=0}}{4!}(x-0)^4 + \frac{\overbrace{f^{(5)}(0)}^{=1}}{5!}(x-0)^5$$

$$\Rightarrow \boxed{T_{5,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

The Maclaurin polynomials contain only odd powers of x, which is due to the fact that $\sin x$ is an odd function!

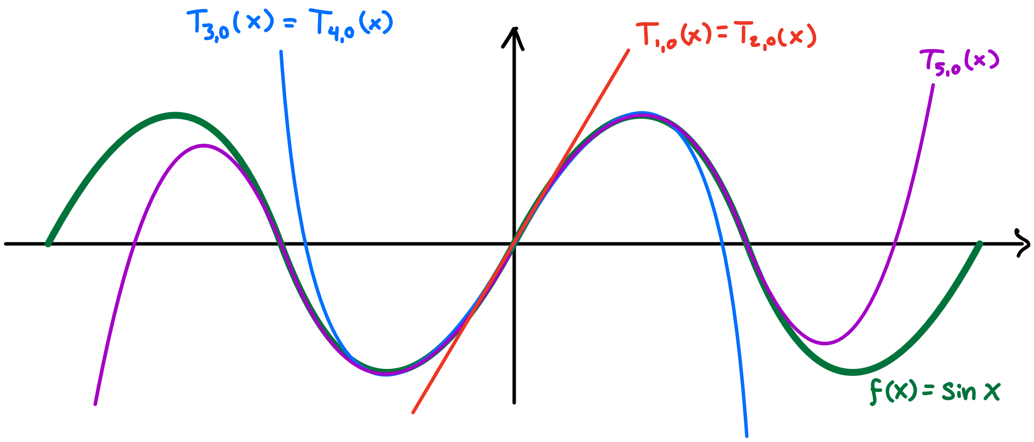
Lower-degree approximations:

$$T_{1,0}(x) = x$$

$$T_{2,0}(x) = x$$

$$T_{3,0}(x) = x - \frac{x^3}{3!}$$

$$T_{4,0}(x) = x - \frac{x^3}{3!}$$



The Maclaurin polynomials for $\cos x$ follow a similar pattern, except involve only even powers (which is due to the fact that $\cos x$ is an even function!)

For $f(x) = \cos(x)$:

$$T_{1,0}(x) = 1$$

$$T_{2,0}(x) = T_{3,0}(x) = 1 - \frac{x^2}{2!}$$

$$T_{4,0}(x) = T_{5,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$