Taylor's Inequality

Recall that the n^{th}-degree Taylor polynomial for $f(x)$ centred at $x = a$ is

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Since $T_{n,a}(x) \approx f(x)$ for x near a, we can use $T_{n,a}(x)$ for approximations!

Ex: Use $T_{2,1}(x)$ for $f(x) = \sqrt{x}$ to approximate $\sqrt{1.1}$.

Solution: In an earlier example, we found

$$T_{2,1}(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

Hence, $\sqrt{1.1} = f(1.1) \approx T_{2,1}(1.1)$

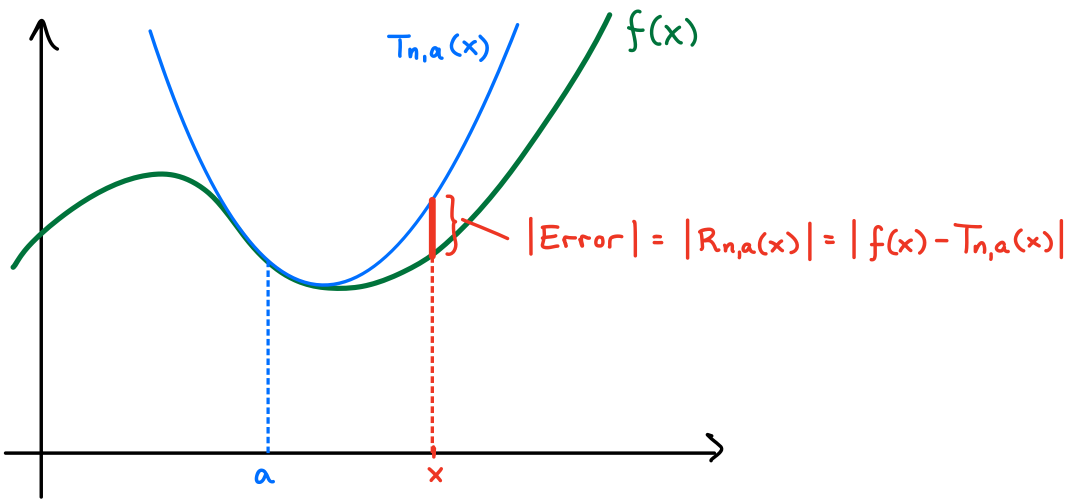
$$= 1 + \frac{1}{2}(1.1-1) - \frac{1}{8}(1.1-1)^2$$

$$= 1 + \frac{1}{2}\underbrace{(0.1)}_{=\frac{1}{10}} - \frac{1}{8}\underbrace{(0.1)^2}_{=\left(\frac{1}{10}\right)^2 = \frac{1}{100}}$$

$$= 1 + \frac{1}{20} - \frac{1}{800} = \boxed{\frac{839}{800} \quad (\text{or } 1.04875)}$$

How good is this approximation?

Definition: The error (or remainder) in using $T_{n,a}(x)$ to approximate $f(x)$ is $\boxed{R_{n,a}(x) = f(x) - T_{n,a}(x)}$



The following result allows us to estimate $|R_{n,a}(x)|$, the magnitude of the error.

Taylor's Inequality

The error in using $T_{n,a}(x)$ to approximate $f(x)$ satisfies

$$\boxed{\left|R_{n,a}(x)\right| \leq \frac{M\,|x-a|^{n+1}}{(n+1)!}}$$

where $\left|f^{(n+1)}(t)\right| \leq M$ for all t between a and x.

<u>Ex</u>: Let $f(x)=\sqrt{x}$. Estimate the size of the error when $T_{2,1}(1.1) = 1.04875$ is used to approximate $f(1.1)=\sqrt{1.1}$.

<u>Solution:</u> By Taylor's inequality,

$a=1$ $n=2$

$$|\text{Error}| = \left|R_{2,1}(x)\right| \leq \frac{M\,|x-1|^{2+1}}{(2+1)!} = \frac{M|x-1|^3}{3!}$$

When $x = 1.1$, we get

$$|R_{2,1}(1.1)| \leq \frac{M\,|1.1 - 1|^3}{3!} = \frac{M\,(0.1)^3}{6} = \frac{M}{6000}$$

To find M, we look at

$$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8x^{5/2}}.$$

On $[1, 1.1]$, we have

We consider all values between $a = 1$ and $x = 1.1$

This is our M!

$$|f'''(x)| = \frac{3}{8x^{5/2}} \leq \frac{3}{8 \cdot 1^{5/2}} = \frac{3}{8}$$

Thus,

$$\boxed{|\text{Error}| = |R_{2,1}(x)| \leq \frac{M}{6000} = \frac{3/8}{6000} = \frac{1}{16000} = 0.0000625.}$$

[For reference, $\sqrt{1.1} \approx 1.0488$, which is indeed quite close to our estimate $T_{2,1}(1.1) = 1.04875$.]

Ex: Use $T_{5,0}(x)$ for $f(x) = \sin x$ to approximate $\sin 1$, then find an upper bound on the error in this approximation.

Solution: Earlier, we showed that

$$T_{5,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Hence,

$$\sin 1 = f(1) \approx T_{5,0}(1)$$

$$= 1 - \frac{1}{6} + \frac{1}{120} = \boxed{\frac{101}{120}}$$

For the error, Taylor's inequality states that

$$|\text{Error}| = |R_{5,0}(x)| \leq \frac{M|x-0|^{5+1}}{(5+1)!} = \frac{M \cdot x^6}{6!}$$

and hence for $x = 1$, we get

$$|R_{5,0}(1)| \leq \frac{M \cdot (1)^6}{6!} = \frac{M}{720}$$

To find M, we look at $f^{(6)}(x) = -\sin x$ for $x \in [0,1]$.

We have

This is our M.

$$|f^{(6)}(x)| = |-\sin x| \leq 1 \quad \text{for } x \in [0,1].$$

[<u>Note:</u> We could have let $M = \sin 1$ — the maximum of $|f^{(6)}(x)|$ for $x \in [0,1]$ — but this would be a little silly, since $\sin 1$ is what we're approximating!]

Thus,

$$\boxed{|\text{Error}| = |R_{5,0}(1)| \leq \frac{M}{720} = \frac{1}{720}}$$

We can now use this error bound to find a range of possible values for $\sin 1$:

$$|R_{2,1}(1)| \leq \frac{1}{720} \Rightarrow |f(1) - T_{5,0}(1)| \leq \frac{1}{720}$$

$$\Rightarrow \frac{-1}{720} \leq \sin 1 - \frac{101}{120} \leq \frac{1}{720}$$

$$\Rightarrow \quad \frac{101}{120} - \frac{1}{720} \leq \sin 1 \leq \frac{101}{120} + \frac{1}{720}$$

$$\Rightarrow \quad \underbrace{\frac{605}{720}}_{\approx 0.8403} \leq \sin 1 \leq \underbrace{\frac{607}{720}}_{\approx 0.8431}$$

Actual value

≈ 0.8415