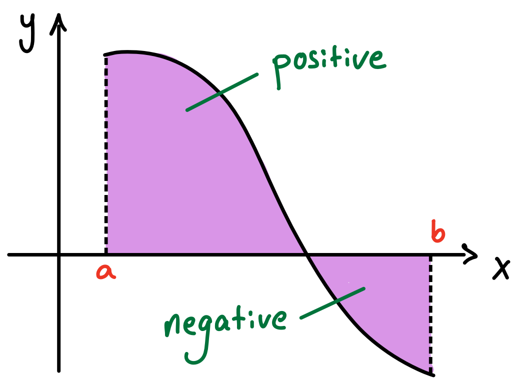
Welcome to MATH 118!

Brief Review from MATH 116

The definite integral $\int_a^b f(x)\,dx$ represents the (signed) area under the graph of $y = f(x)$ from $x = a$ to $x = b$ and above the x-axis.



We can evaluate definite integrals using the Fundamental Theorem of Calculus (FTC)

FTC (Part I): If f is continuous, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is antiderivative of f.

Recall the following elementary antiderivatives:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Ex: $$\int_0^1 (2x+1)\,dx = \left[x^2+x\right]_0^1 = (1^2+1)-(0^2+0) = \boxed{2.}$$

For more complicated integrals, more work is needed!

Part II of the FTC says that derivatives and integrals are inverse operations

FTC (Part II): If f is continuous, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

More generally,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Ex: What is $\displaystyle\frac{d}{dx}\int_{x^2+1}^{3x+1} e^{\sqrt{t}}\,dt$?

Solution: $$\frac{d}{dx}\int_{x^2+1}^{3x+1} e^{\sqrt{t}}\,dt = e^{\sqrt{3x+1}}(3x+1)' - e^{\sqrt{x^2+1}}(x^2+1)'$$

$$= \boxed{3e^{\sqrt{3x+1}} - 2xe^{\sqrt{x^2+1}}}$$

Substitution Rule

Idea: let $u =$ ______ , then $du =$ ______ dx

Replace u & dx in the integral and evaluate.

Tip: Some good choices for u include

$u =$ function raised to an ugly power

$u =$ function inside sin, cos, ln, e^{-}, etc...

$u =$ function whose derivative is also present

Ex: $\int e^{\sin x} \cos x \, dx$

Solution: Let $u = \sin x$, so $du = \cos x \, dx$. We have

$$\int e^{\sin x} \cos x \, dx = \int e^u \, du$$

$$= e^u + C = \boxed{e^{\sin x} + C}$$

Ex: $\displaystyle\int_0^1 \frac{x^3}{1+3x^4}\,dx$

Solution: Let $u = 1+3x^4$, so $du = 12x^3 dx$ $\left(\text{or } dx = \frac{1}{12x^3}du\right)$.

Option 1: Convert the bounds

$$x=0 \Rightarrow u = 1+3(0)^4 = 1$$

$$x=1 \Rightarrow u = 1+3(1)^4 = 4$$

$$\int_0^1 \frac{x^3}{1+3x^4}\,dx = \int_1^4 \frac{\cancel{x^3}}{u}\cdot\frac{1}{12\cancel{x^3}}\,du$$

$$= \frac{1}{12}\int_1^4 \frac{1}{u}\,du$$

$$= \frac{1}{12}\Big[\ln|u|\Big]_1^4 = \frac{1}{12}\left(\ln 4 - \ln 1\right) = \boxed{\frac{\ln 4}{12}}$$

Option 2: Don't convert the bounds (but go back to x!)

$$\int_0^1 \frac{x^3}{1+3x^4}\,dx = \int_{x=0}^{x=1} \frac{\cancel{x^3}}{u}\cdot\frac{1}{12\cancel{x^3}}\,du$$

$$= \frac{1}{12}\int_{x=0}^{x=1} \frac{1}{u}\, du$$

Must write "x=" to indicate that these are NOT bounds with respect to u!

$$= \frac{1}{12}\Big[\ln|u|\Big]_{x=0}^{x=1}$$

$$= \frac{1}{12}\Big[\ln|1+3x^4|\Big]_{x=0}^{x=1} = \boxed{\frac{\ln 4}{12}}$$

Ex: $\displaystyle\int \frac{z+6}{\sqrt{z+5}}\, dz$

so $z = u-5$

Solution: Let $u = z+5$, so $du = dz$. We have

$$\int \frac{z+6}{\sqrt{z+5}}\, dz = \int \frac{(u-5)+6}{\sqrt{u}}\, du = \int \frac{u-1}{\sqrt{u}}\, du$$

$$= \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$$

$$= \int \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C$$

$$= \boxed{\frac{2}{3}(z+5)^{3/2} - 2(z+5)^{1/2} + C}$$

Integration by Parts (IBP)

The IBP formula states that

$$\boxed{\int u\,dv = uv - \int v\,du}$$

or, for definite integrals:

$$\boxed{\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du}$$

Ex: $\displaystyle\int_0^{\pi} x\cos x\,dx$

Solution: Let $u = x$ $\quad v = \sin x$

$du = dx \quad dv = \cos x\,dx$

We have $\displaystyle\int_0^{\pi} x\cos x\,dx = \Big[x\sin x\Big]_0^{\pi} - \int_0^{\pi} \sin x\,dx$

$$= \underbrace{\cancel{\pi\sin\pi}}_{=0} - \underbrace{\cancel{0\sin 0}}_{=0} - \Big[-\cos x\Big]_0^{\pi}$$

$$= \cos(\pi) - \cos(0)$$

$$= \boxed{-2}$$

Tip: Let u be the first function on the list below that appears in your integral, then let the remainder of the integrand be dv.

Logs
Inverse trig
Algebraic (i.e., x^n or polynomials)
Trig.
Exponential

Ex: $\displaystyle\int x^2 e^x \, dx$

Solution: Let $u = x^2$ $\quad v = e^x$

$du = 2x\,dx \quad dv = e^x\,dx$

$$\int x^2 e^x \, dx = x^2 e^x - \underbrace{\int 2x e^x \, dx}_{\text{use IBP again!}}$$

$u = 2x \quad v = e^x$

$du = 2\,dx \quad dv = e^x dx$

$$= x^2 e^x - \left[2x e^x - \int 2 e^x \, dx \right]$$

$$= \boxed{x^2e^x - 2xe^x + 2e^x + C}$$

Ex: $\int \arcsin(x)\,dx$

Solution: Let $u = \arcsin(x)$ $\quad v = x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \quad dv = dx$

$$\Rightarrow \int \arcsin(x)\,dx = x\cdot\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

u-sub!

$u = 1-x^2$

$\Rightarrow du = -2x\,dx$

(so $dx = \frac{-1}{2x}\,du$)

$$= x\cdot\arcsin(x) - \int \frac{x}{\sqrt{u}}\cdot\left(\frac{-1}{2x}\,du\right)$$

$$= x\cdot\arcsin(x) + \frac{1}{2}\int u^{-1/2}\,du$$

$$= x\cdot\arcsin(x) + \cancel{\frac{1}{2}}\left[\frac{u^{1/2}}{\cancel{1/2}}\right] + C$$

$$= \boxed{x\cdot\arcsin(x) + \sqrt{1-x^2} + C}$$

Additional Exercises:

A good challenge!

1. $\int x^3\sqrt{1+x^2}\,dx$ 2. $\int \ln x\,dx$ 3. $\int \cos(\ln x)\,dx$

Solutions:

So $x^2 = u-1$

1. Let $u = 1+x^2$, so $du = 2x\,dx$ (hence $dx = du/2x$)

$$\int x^3\sqrt{1+x^2}\,dx = \int x^3\sqrt{u}\cdot\left(\frac{du}{2x}\right)$$

$$= \frac{1}{2}\int x^2 u^{1/2}\,du$$

$$= \frac{1}{2}\int (u-1)\,u^{1/2}\,du$$

$$= \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C$$

$$= \boxed{\frac{(1+x^2)^{5/2}}{5} - \frac{(1+x^2)^{3/2}}{3} + C}$$

2. $\displaystyle\int \ln x \, dx$

IBP: $u = \ln x \qquad v = x$

$du = \frac{1}{x}dx \qquad dv = 1\,dx$

$$= x\ln x - \int x \cdot \left(\frac{1}{x}\right) dx$$

$$= x\ln x - \int 1\,dx = \boxed{x\ln x - x + C}$$

3. $\displaystyle\int \cos(\ln x)\,dx$

Let $w = \ln x$ (so $x = e^w$)

$dw = \frac{1}{x}dx \quad (dx = x\,dw = e^w dw)$

$$= \int e^w \cos(w)\,dw$$

IBP: $u = \cos w \qquad v = e^w$

$du = -\sin w\,dw \qquad dv = e^w\,dw$

$$= e^w \cos w - \int e^w(-\sin w)\,dw$$

$$= e^w \cos w + \int e^w \sin w\,dw$$

IBP: $u = \sin w \qquad v = e^w$

$du = \cos w\,dw \qquad dv = e^w dw$

$$= e^w \cos w + \left[e^w \sin w - \int e^w \cos w\,dw\right]$$

We have shown that

$$\int e^w \cos w \, dw = e^w \cos w + e^w \sin w - \int e^w \cos w \, dw$$

$$\Rightarrow 2\int e^w \cos w \, dw = e^w \cos w + e^w \sin w + C$$

$$\Rightarrow \int e^w \cos w \, dw = \frac{1}{2}\left[e^w \cos w + e^w \sin w\right] + C$$

Putting it all together, we have

$$\int \cos(\ln x)\, dx = \int e^w \cos w \, dw$$

$$= \frac{1}{2}\left[e^w \cos w + e^w \sin w\right] + C$$

$$= \frac{1}{2}\left[\underbrace{e^{\ln x}}_{=x} \cos(\ln x) + \underbrace{e^{\ln x}}_{=x} \sin(\ln x)\right] + C$$

$$= \boxed{\frac{x}{2}\left[\cos(\ln x) + \sin(\ln x)\right] + C}$$

Phew!