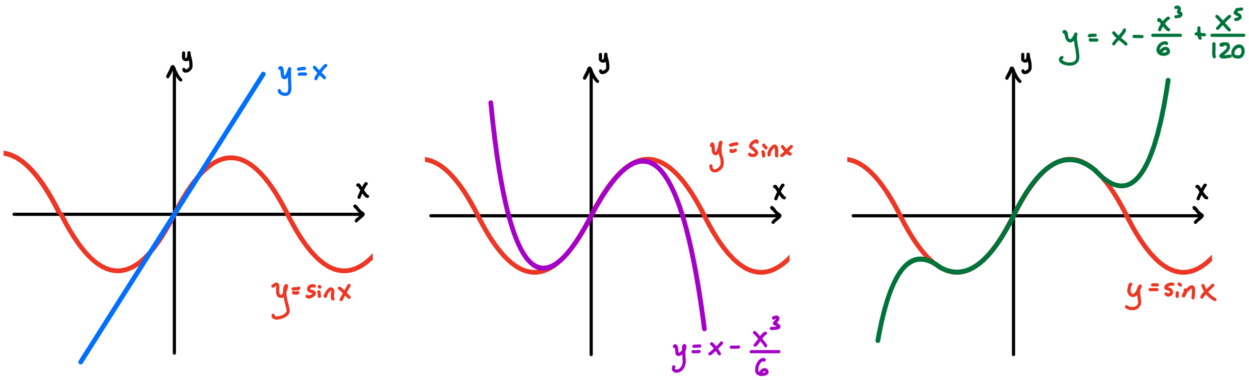
Sequences, Series, and Approximation

We're now entering our next major unit of the course on sequences, series, and approximation. Our goal over the next many weeks is to learn how to approximate complicated functions using sequences of simpler functions — polynomials!



Approximating functions in this way has tons of applications throughout math, physics, and engineering. But before we can study sequences of functions, we'll need to understand sequences of numbers.

§10.1, 10.8 – Sequences and their Limits

An sequence is a list of numbers with a definite order: $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ $(a_i \in \mathbb{R})$

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ may also be written as $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

A sequence may have each term specified explicitly (in terms of n):

e.g. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

e.g. $\left\{\sqrt{n+2}\right\}_{n=3}^{\infty}$: $\sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$

e.g. $\left\{(-1)^n\right\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \ldots$

Or a sequence may be defined recursively (in terms of previous terms):

e.g. $a_1 = \frac{1}{2}$ and $a_{n+1} = \frac{1+a_n}{2}$ $(n \geq 1)$ describes the sequence $a_1 = \frac{1}{2}$

$$a_2 = \frac{1+1/2}{2} = \frac{3}{4}$$

$$a_3 = \frac{1+3/4}{2} = \frac{7}{8}$$

$$a_4 = \frac{1+7/8}{2} = \frac{15}{16}$$

$$\vdots$$

e.g. The Fibonacci sequence is defined by

$$\boxed{a_1 = 1, \quad a_2 = 1, \quad \text{and} \quad a_{n+2} = a_{n+1} + a_n \quad (n \geq 1)}$$

We have $a_1 = 1$,

$$a_2 = 1$$

$$a_3 = a_2 + a_1 = 1 + 1 = 2$$

$$a_4 = a_3 + a_2 = 2 + 1 = 3$$

$$\vdots$$

$(1, 1, 2, 3, 5, 8, 13, 21, \ldots)$

We have encounted sequences already in MATH 116/118!

Ex: Newton's method uses recursive sequences to approximate roots!

$$x_0 = \text{Some initial guess}$$

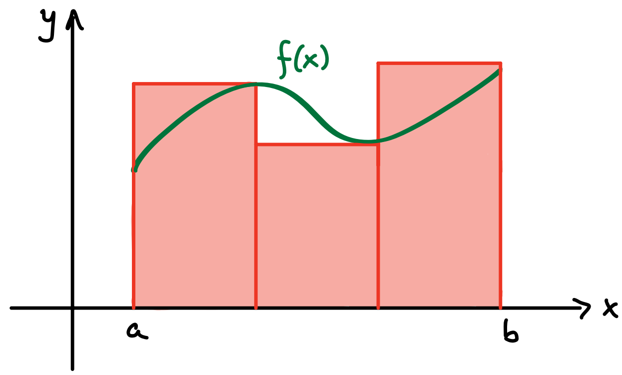
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 1$$

Ex: Riemann sums form a sequence of approximations to the area under a curve:

a_1 = approximation using 1 rectangular area

a_2 = approximation using 2 rectangular areas

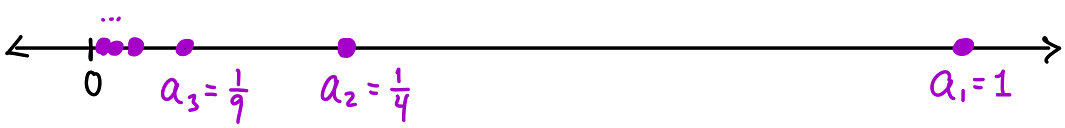
$\vdots$



Visualizing Sequences

We can visualize a sequence on the real number line!

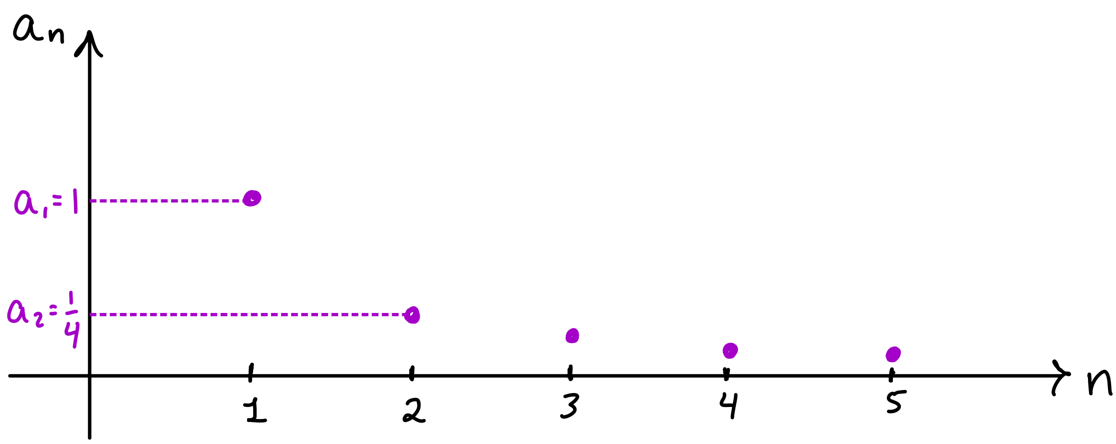
Ex: $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$: $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$



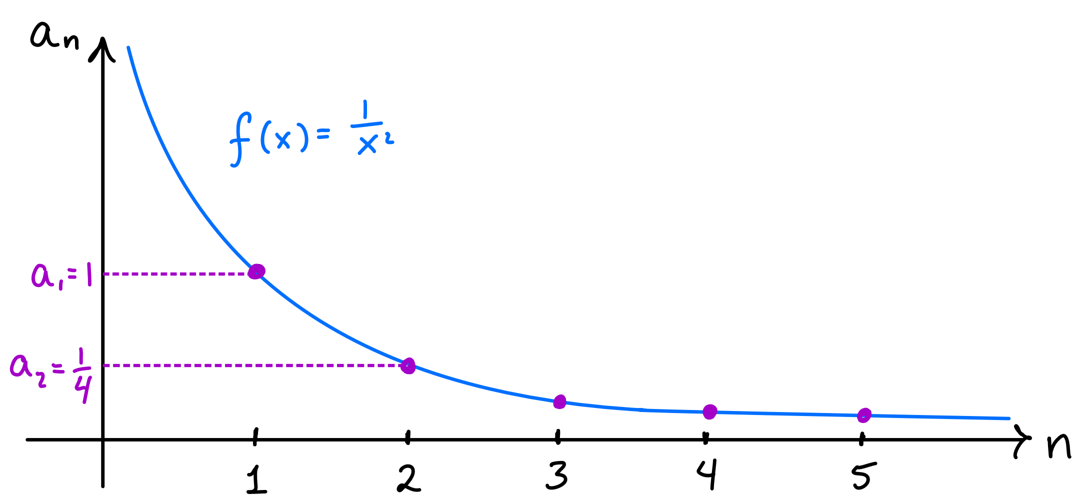
(a bit crowded!)

Or we can visualize a sequence in 2D!

Ex: $\left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty}$: $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$



This is similar to the graph of $f(x) = \frac{1}{x^2}$, but the domain is restricted to the set of natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

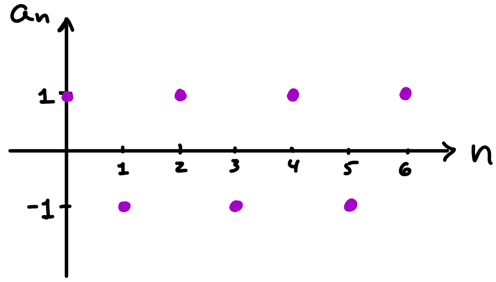


Ex: Create a 2D plot of the following sequences:

(a) $\left\{(-1)^n\right\}_{n=1}^{\infty}$ (b) $\left\{1+\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

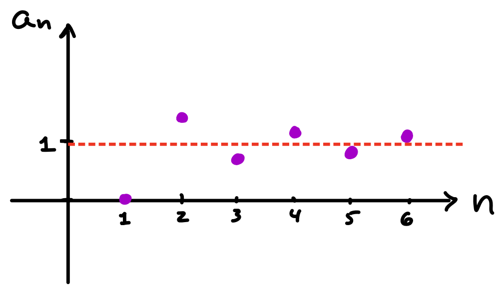
Solution:

(a) $\left\{(-1)^n\right\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \ldots$



(b) $\left\{1+\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$: $1-\frac{1}{1}, 1+\frac{1}{2}, 1-\frac{1}{3},$

$1-\frac{1}{4}, 1+\frac{1}{5}, \ldots$



Limits of Sequences

From the plots in the last example, it seems that

- $\left\{\frac{1}{n^2}\right\}$ converges to 0 as n tends to infinity.
- $\left\{1+\frac{(-1)^n}{n}\right\}$ converges to 1 as n tends to infinity.

But let's not get ahead of ourselves — we haven't even defined yet what it means for a sequence to converge to a number L!

> **Q:** what does it mean for a sequence $\{a_n\}$ to converge to a value L as n tends to ∞?

Attempt 1: "a_n gets closer and closer to L as n becomes large"

Good start... $\{\frac{1}{n^2}\}$ gets closer and closer to 0, however it also gets closer and closer to -1, $-2/3$, etc.

What's special about $L=0$?

Attempt 2: "a_n gets infinitely close to L as n becomes large"

Better, but still imprecise. What does "infinitely close" mean?

"closer than any given value"

And how do we measure "closeness"?

Use $|a_n - L|$ to measure distance between a_n & L.

Attempt 3: "For every $\varepsilon > 0$, eventually the distance $|a_n - L|$ is less than ε."

Very close, but what do we mean by "eventually"?

For all n after some cutoff, N.

Attempt 4: "For every $\varepsilon > 0$, there exists a number N such that $|a_n - L| < \varepsilon$ for all $n > N$."

This is the definition we want!

Definition: A real number L is the limit of $\{a_n\}$ if:

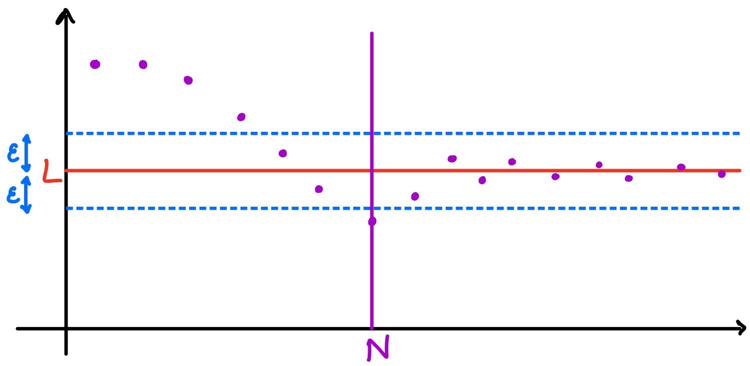
For all $\varepsilon > 0$, there exists a number N such that $|a_n - L| < \varepsilon$ for all $n > N$.

In this case we say $\{a_n\}$ converges to L and write

$$\boxed{\lim_{n \to \infty} a_n = L.}$$

If instead no such L exists, we say $\{a_n\}$ diverges.

For every $\varepsilon > 0$, we need to find an N after which all a_n's are no more than ε away from L.



For instance, suppose we wished to use the above definition to show that

$$\lim_{n \to \infty} \frac{1}{n^2} = 0.$$

If $\varepsilon = \frac{1}{100}$ is given, we must find a point N after which $|a_n - L| = \left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2}$ is less than $\varepsilon = \frac{1}{100}$

Note that

$$\frac{1}{n^2} < \frac{1}{100} \quad \Rightarrow \quad 100 < n^2 \quad \Rightarrow \quad n > 10.$$

So $|a_n - L| < \frac{1}{100}$ will be true for all n after $N = 10$.

We have successfully found an N for this ε!

Note: The above is **NOT** a proof that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

So far we have considered just $\varepsilon = \frac{1}{100}$. Can we find N that works when $\varepsilon = \frac{1}{1000}$? $\varepsilon = \frac{1}{1\,000\,000}$? We must generalize our argument to work for **every** ε!

Ex: Use the definition of the limit of a sequence to prove the following:

(a) $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

Solution:

Rough Work: Given $\varepsilon > 0$, we want

$$|a_n - L| = \left|\frac{1}{n^2} - 0\right| < \varepsilon$$

$$\Rightarrow \frac{1}{n^2} < \varepsilon$$

$$\Rightarrow \frac{1}{\varepsilon} < n^2$$

$$\Rightarrow \frac{1}{\sqrt{\varepsilon}} < n$$

Aha! The inequality will work for n after $N = \frac{1}{\sqrt{\varepsilon}}$!

Now we write our proof!

Proof that $\lim_{n\to\infty} \frac{1}{n^2} = 0$:

Let $\varepsilon > 0$ be given. Let $N = \frac{1}{\sqrt{\varepsilon}}$. In this case, if $n > N$, we

$\left(n > N \Rightarrow \frac{1}{n} < \frac{1}{N}\right)$

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \frac{1}{N^2}$$
$$= \frac{1}{(1/\sqrt{\varepsilon})^2}$$
$$= \frac{1}{1/\varepsilon} = \varepsilon. \qquad \blacksquare$$

(b) $\lim_{n\to\infty} \frac{n+4}{n+9} = 1$

Rough work: Given $\varepsilon > 0$, We want

$$|a_n - L| = \left|\frac{n+4}{n+9} - 1\right| = \left|\frac{n+4-(n+9)}{n+9}\right|$$
$$= \left|\frac{-5}{n+9}\right| = \frac{5}{n+9} < \varepsilon$$
$$\Rightarrow \frac{5}{\varepsilon} < n+9$$
$$\Rightarrow \frac{5}{\varepsilon} - 9 < n$$

Okay, let's pick $N = \frac{5}{\varepsilon} - 9$.

Proof that $\lim_{n\to\infty} \frac{n+4}{n+9} = 1$:

Let $\varepsilon > 0$ be given. Let $N = \frac{5}{\varepsilon} - 9$. In this case,

if $n > N$, then

$\left(n > N \Rightarrow n+9 > N+9 \Rightarrow \frac{5}{n+9} < \frac{5}{N+9}\right)$

$$\left|\frac{n+4}{n+9} - 1\right| = \frac{5}{n+9} < \frac{5}{N+9}$$

$$= \frac{5}{\left(\frac{5}{\varepsilon} - \cancel{9}\right) + \cancel{9}}$$

$$= \frac{5}{5/\varepsilon} = \varepsilon \quad \blacksquare$$

(c) $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

<u>Rough work</u>: Given $\varepsilon > 0$, we want

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n - (2n+1)}{2(2n+1)}\right| = \left|\frac{-1}{4n+2}\right| = \frac{1}{4n+2} < \varepsilon$$

$$\Rightarrow \frac{1}{\varepsilon} < 4n+2$$

$$\Rightarrow \frac{1}{\varepsilon} - 2 < 4n$$

$$\Rightarrow n > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)$$

Okay, let's pick $N = \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)$.

Proof that $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$:

Let $\varepsilon > 0$ be given. Let $N = \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)$.

In this case, if $n > N$, then

$$\left| \frac{n}{2n+1} - \frac{1}{2} \right| = \frac{1}{4n+2} < \frac{1}{4N+2} = \frac{1}{\cancel{4}\left[\cancel{\frac{1}{4}}\left(\frac{1}{\varepsilon}-2\right)\right]+2}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon} - \cancel{2}\right) + \cancel{2}}$$

$$= \varepsilon. \quad \blacksquare$$

(d) $\lim_{n\to\infty} \frac{\sin(n)}{n^2+6} = 0.$

Solution:

Rough work: We want

$$\left| \frac{\sin(n)}{n^2+6} \right| = \frac{|\sin(n)|}{n^2+6} \leq \frac{1}{n^2+6} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < n^2+6$$

$$\Rightarrow n^2 > \frac{1}{\varepsilon} - 6$$

$$\Rightarrow n > \sqrt{\frac{1}{\varepsilon} - 6}$$

Okay, let's pick $N = \sqrt{\frac{1}{\varepsilon} - 6}$.

Proof that $\lim_{n\to\infty} \frac{\sin(n)}{n^2+6} = 0$:

Let $\varepsilon > 0$ be given. Let $N = \sqrt{\frac{1}{\varepsilon} - 6}$. In this case, if $n > N$, then

$$\left| \frac{\sin(n)}{n^2+6} \right| \leq \frac{1}{n^2+6} < \frac{1}{N^2+6} = \frac{1}{\left(\sqrt{\frac{1}{\varepsilon}-6}\right)^2+6} = \frac{1}{\left(\frac{1}{\varepsilon}-\cancel{6}\right)+\cancel{6}} = \varepsilon$$

$\blacksquare$

In practice, there are easier ways to calculate limits so only use the definition of convergence if asked to!

Limits of Sequences in Practice

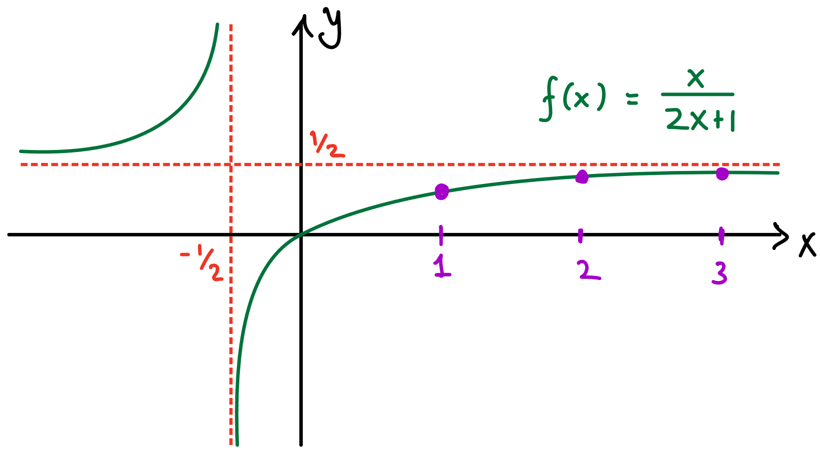
Let's again consider $\lim_{n\to\infty} \frac{n}{2n+1}$.

Trick: consider the function

$$f(x) = \frac{x}{2x+1} \quad \text{where } x \in \mathbb{R},\ x \neq \frac{-1}{2}.$$

From limits in MATH 116, we know that

$$\lim_{x\to\infty} \frac{x}{2x+1} = \lim_{x\to\infty} \frac{\cancel{x}\cdot 1}{\cancel{x}\left(2+\frac{1}{x}\right)} = \frac{1}{2+0} = \frac{1}{2}$$



Since $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$ is contained within the graph of $f(x) = \frac{x}{2x+1}$, this sequence must also have a limit of $\boxed{\frac{1}{2}}$!

More generally, we have the following:

> If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, where $L \in \mathbb{R}$ or $L = \pm\infty$, then $\lim_{n\to\infty} a_n = L$ also.

By thinking of $\lim_{n\to\infty} a_n$ (the limit of a sequence) as $\lim_{x\to\infty} f(x)$ (the limit of a function of x, $x \in \mathbb{R}$), we get all of our usual limit laws...

Limit Laws: If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$,

where $A, B \in \mathbb{R}$ (i.e., these limits exist), then

(i) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

(ii) $\lim_{n\to\infty} a_n \cdot b_n = A \cdot B$

(iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$.

(iv) $\lim_{n\to\infty} c \cdot a_n = c \cdot A$, $c \in \mathbb{R}$

(v) $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(A)$ if f is continuous.

As well as the squeeze theorem...

Squeeze Theorem:

Since we only care what happens as $n \to \infty$, we only need $a_n \leq b_n \leq c_n$ for large n!

If $a_n \leq b_n \leq c_n$ (eventually) and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$ also.

and even L'Hopital's rule!

Examples:

(a) $$\lim_{n\to\infty} \frac{n^3+n}{4n^3+1} = \lim_{n\to\infty} \frac{\cancel{n^3}(1+1/n^2)}{\cancel{n^3}(4+1/n^3)} = \frac{1+0}{4+0} = \boxed{\frac{1}{4}}$$

(b) $$\lim_{n\to\infty} \ln\left(\frac{n+1}{\sqrt{n^2+1}}\right) = \ln\left(\lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{\sqrt{n^2}}\sqrt{1+1/n^2}}\right)$$

Bring limit inside since $\ln(x)$ is cts!

$$= \ln\left(\frac{1+0}{\sqrt{1+0}}\right) = \ln(1) = \boxed{0}.$$

(c) $$\lim_{n\to\infty} \underbrace{\left(\sqrt{n+1}-\sqrt{n}\right)}_{\text{"}\infty-\infty\text{"} \Rightarrow \text{ indeterminate!}} \cdot \frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \boxed{0}$$

"$\frac{1}{\infty}$"

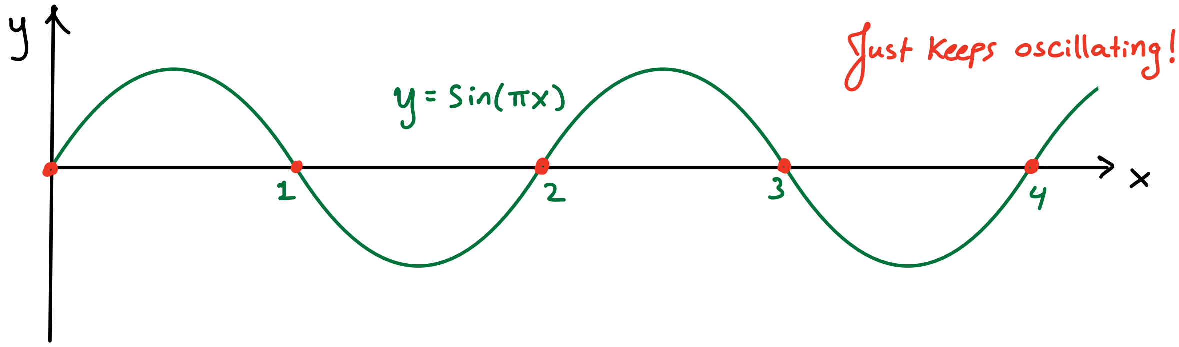
(e) $$\lim_{n\to\infty} \frac{\cos(n^3+n+1)}{n^2}$$

Note: $-1 \leq \cos(n^3+n+1) \leq 1 \Rightarrow \underbrace{\frac{-1}{n^2}}_{\to 0} \leq \frac{\cos(n^3+n+1)}{n^2} \leq \underbrace{\frac{1}{n^2}}_{\to 0}$

$\therefore \lim_{n\to\infty} \frac{\cos(n^3+n+1)}{n^2} = \boxed{0}$ by the squeeze theorem!

(f) $$\lim_{n\to\infty} \sin(\pi n)$$

Note that $\lim_{x \to \infty} \sin(\pi x)$ DNE :



However... $\sin(\pi n) = 0$ for all $n \in \mathbb{N}$! Thus,

$$\lim_{n \to \infty} \sin(\pi n) = \boxed{0.}$$