§15.2 – Separable Differential Equations

A first order DE is said to be separable if it can be written as

$$\boxed{\frac{dy}{dx} = g(x)\cdot h(y)}$$

That is, we can factor the non-derivative terms as a product of an x-part and a y-part.

e.g., $\frac{dy}{dx} = \frac{x}{y} \quad \left(= x\cdot\frac{1}{y}\right) \implies$ Separable

$\frac{dy}{dx} = x+2y \,, \quad \frac{dy}{dx} = \sin(xy) \implies$ Non-separable

To solve: Split up the differential, separate the x's and y's, and integrate!

$$\frac{dy}{dx} = g(x)\,h(y) \implies \frac{1}{h(y)}\,dy = g(x)\,dx$$

$$\implies \int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Ex: Solve $\frac{dy}{dx} = \frac{x}{y}$

Solution: $\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$

$$\Rightarrow \int y\,dy = \int x\,dx$$

Now isolate for y (if possible!)

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

For convenience, let's write $D = 2C$.

$$\Rightarrow y^2 = x^2 + 2C$$

$$\Rightarrow \boxed{y = \pm\sqrt{x^2 + D}, \quad D \in \mathbb{R}}$$

Note: When $D = 1$ we get $y = \sqrt{x^2+1}$, which is the solution we verified in our last example!

If we are told the value of the unknown function y at some given x, we can figure out the arbitrary constant. A DE together with point (x_0, y_0) is called an <u>initial value problem (IVP)</u>.

Ex: Solve the IVP $\dfrac{dy}{dx} = \dfrac{x\ln x}{3y^2}$, $y(1) = 0$.

Solution: $$\int 3y^2\,dy = \int x\ln x\,dx \leftarrow \text{IBP!}$$

$u = \ln x$ | $v = \frac{x^2}{2}$

$du = \frac{1}{x}dx$ | $dv = x\,dx$

$$\Rightarrow y^3 = \frac{x^2}{2}\ln x - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx$$

$$\Rightarrow y^3 = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

using $y(1)=0$, we have

$$0^3 = \underbrace{\tfrac{1}{2}\ln 1}_{=0} - \tfrac{1}{4} + C,$$

hence $C = 1/4$.

$$\Rightarrow y^3 = \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{1}{4}$$

$$\Rightarrow \boxed{y = \sqrt[3]{\frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{1}{4}}}$$

Ex: Find the general solution to $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+2y^2}$.

Solution: $$\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx \qquad (*)$$

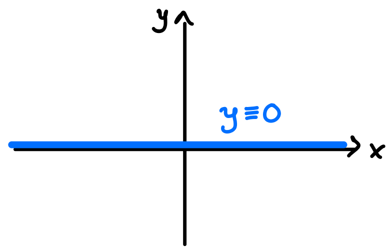
$$\Rightarrow \int \left(\frac{1}{y} + 2y\right)dy = \sin x + C$$

$$\Rightarrow \underbrace{\ln|y| + y^2 = \sin x + C}_{\text{Can't solve for } y = f(x) \text{, so we'll stop here.}}$$

BUT WAIT — there may be another solution!

We divided by y in the first step ⊛, but what if $y \equiv 0$?

The notation "$y \equiv 0$" means "y is identically / constantly 0."



We'll need to check the $y \equiv 0$ case separately!

If $y \equiv 0$ then $\frac{dy}{dx} = 0$ and $\frac{y\cos x}{1+2y^2} = \frac{0 \cdot \cos x}{1+0} = 0$

So yes, $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$ when $y \equiv 0$!

General Solution:

$$y \equiv 0 \quad \text{or} \quad \ln|y| + y^2 = \sin x + C, \quad C \in \mathbb{R}$$

Ex: Solve the IVP $y' = xy$, $y(0) = 2$.

Solution: $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx$ if $y \not\equiv 0$

(but in this case $y(0)=2$, so $y \equiv 0$ isn't possible!)

$$\Rightarrow \int \frac{dy}{y} = \int x\,dx$$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

$$\Rightarrow y = \pm e^C \cdot e^{\frac{x^2}{2}} = Ae^{\frac{x^2}{2}}$$

For convenience, write $A = \pm e^C$

Using the initial condition $y(0)=2$, we have

$$2 = Ae^{\frac{0^2}{2}} = A \cdot 1 \Rightarrow A = 2.$$

Thus, $\boxed{y = 2e^{\frac{x^2}{2}}}$

Additional Exercises

1. Find the general solution to $\frac{dy}{dx} = y^2$.

2. Solve the IVP $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$, $y(0) = -1$

Solutions

We'll separately check $y \equiv 0$.

1. $\frac{dy}{dx} = y^2 \Rightarrow \int \frac{1}{y^2}\,dy = \int 1 \cdot dx$

$$\Rightarrow \frac{-1}{y} = x + C$$

$$\Rightarrow \frac{1}{y} = -x - C$$ Let $D = -C$

$$\Rightarrow y = \frac{1}{D-x}, \quad D \in \mathbb{R}$$

Check $y \equiv 0$: $\frac{dy}{dx} = 0$, $y^2 = 0$ (equal!)

$\Rightarrow y \equiv 0$ is a solution!

Thus, $\boxed{y \equiv 0 \quad \text{or} \quad y = \frac{1}{D-x}, \quad D \in \mathbb{R}}$

2. $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2} \Rightarrow \int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$

$$\Rightarrow y^2 - 2y = x^3 + 2x^2 + 2x + C$$

Since $y(0) = -1$, we have

$$\underbrace{(-1)^2 - 2(-1)}_{=3} = \underbrace{0^3 + 2(0)^2 + 2(0)}_{=0} + C,$$

and hence $C = 3$. Thus,

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

Solve for y by completing the square:

$$(y^2 - 2y + 1) - 1 = x^3 + 2x^2 + 2x + 3$$

$$\Rightarrow \quad (y-1)^2 = 1 + (x^3 + 2x^2 + 2x + 3)$$

$$\Rightarrow \quad y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4}$$

$$\Rightarrow \quad y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

However, only

$$\boxed{y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}}$$

satisfies the initial condition $y(0) = -1$.