§ 15.4 – Reducible Second Order Differential Equations

We'll say a second order DE (i.e., one involving only x, y, y', y'') is reducible if it can be transformed into a first order DE. We can then solve the DE using our earlier methods. We'll consider two cases of reducible second order DEs.

Case I: y does not appear

Here we will solve DEs involving only x, y', and y''.

Ex: Solve $xy'' + y' = 8x$

> Idea: Let $y' = v$, so $y'' = v'$. This will reduce the problem to a first order DE involving x and v!

Solution: Let $y' = v$, so $y'' = v'$. We have

$$xy'' + y' = 8x \Rightarrow xv' + v = 8x$$ (Linear! Divide by x!)

$$\Rightarrow \quad v' + \underbrace{\frac{1}{x}}_{P(x)} v = 8$$

We multiply by

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$$

giving us $\underbrace{xv' + v}_{[x^2 v]'} = 8x \Rightarrow [xv]' = 8x$

$$\Rightarrow \quad xv = 4x^2 + C, \quad C \in \mathbb{R}$$

$$\Rightarrow \quad v = 4x + \frac{C}{x}, \quad C \in \mathbb{R}.$$

Recall that $v = y'$, hence we have just shown that

$$y' = 4x + \frac{C}{x}, \quad C \in \mathbb{R}$$

Finally, integrate to get y:

$$y = \int \left(4x + \frac{C}{x}\right) dx$$

$$\Rightarrow \quad \boxed{y = 2x^2 + C\ln|x| + D, \quad C, D \in \mathbb{R}}$$

We now have a two-parameter family of solutions!

Ex: Solve $y'' = \dfrac{y'}{x}$

Solution: Since y does not appear, let $y' = v$, $y'' = v'$.

Then

$$y'' = \frac{y'}{x} \Rightarrow v' = \frac{v}{x}$$

Separable! $\dfrac{dv}{dx} = \dfrac{v}{x}$

Linear! $v' - \frac{1}{x} v = 0.$

Approach 1: Solve as a Separable DE

$$\frac{dv}{dx} = \frac{v}{x} \Rightarrow \int \frac{dv}{v} = \int \frac{dx}{x} \qquad (\text{provided } v \neq 0)$$

$$\Rightarrow \ln|v| = \ln|x| + c$$

$$\Rightarrow |v| = e^{\ln|x| + c} = e^c \cdot |x|$$

$$\Rightarrow v = \pm e^c x$$

Thus, $v = y' = \pm e^c x \Rightarrow y = \dfrac{\pm e^c}{2} x^2 + D$

$$\Rightarrow \underline{y = C_1 x^2 + C_2}$$

$C_2 \in \mathbb{R}$

$C_1 = \dfrac{\pm e^c}{2}$, $C_1 \neq 0$

We must now consider $v = y' \equiv 0$, in which case $y =$ constant; hence the DE becomes $0 = 0$ (which is true!)

Thus, we have

$$\boxed{y = C_1 x^2 + C_2, \quad C_1 \neq 0, \quad C_2 \in \mathbb{R} \quad \underline{\text{or}} \quad y = C_3, \quad C_3 \in \mathbb{R}}$$

We can actually combine these possibilities into one big solution:

we now allow $C_1 = 0$, which lets us get $y = C_2$.

$$\boxed{y = C_1 x^2 + C_2, \quad C_1, C_2 \in \mathbb{R}}$$

Approach 2: Solve as a Linear DE (usually "cleaner"!)

$$v' = \frac{v}{x} \quad \Rightarrow \quad v' - \frac{1}{x} v = 0$$

Multiply by

$$\mu(x) = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$$

Hence $v' - \frac{1}{x} v = 0 \quad \Rightarrow \quad \frac{1}{x} v' - \frac{1}{x^2} v = 0$

$$\Rightarrow \quad \left[\frac{1}{x} v \right]' = 0$$

integrate!

$$\Rightarrow \quad \frac{1}{x} v = C$$

$$\Rightarrow \quad v = Cx, \quad C \in \mathbb{R}$$

Thus,

$$v = \frac{dy}{dx} = Cx \implies \boxed{y = \frac{Cx^2}{2} + D, \quad C, D \in \mathbb{R}}$$

or, by letting $C_1 = C/2$ and $C_2 = D$:

$$\boxed{y = C_1 x^2 + C_2, \quad C_1, C_2 \in \mathbb{R}}$$

Case II: x does not appear

We'll now solve DEs involving only y, y', and y''.

Ex: Solve $y'' = \frac{y'}{y^2}$ given $y(0) = 2$, $y'(0) = -1/2$.

Idea: We'll again let $y' = \frac{dy}{dx} = v$, but this time, to avoid introducing any x's to the DE, we'll write

$$y'' = \frac{dv}{dx} = \frac{dv}{dy} \cdot \underbrace{\frac{dy}{dx}}_{=v} = v\frac{dv}{dy}$$

Step 1: Start by writing $y' = v$ and $y'' = v\frac{dv}{dy}$

In our example: $y'' = \dfrac{y'}{y^2} \Rightarrow v\dfrac{dv}{dy} = \dfrac{v}{y^2}$

Note: You should now have a first-order DE involving just y's and v's!

Step 2: Solve the new DE for v as a function of y.

In our example:

$$v\frac{dv}{dy} = \frac{v}{y^2} \Rightarrow \frac{v\,dv}{v} = \frac{dy}{y^2}$$

(Separable!)

[provided $v \not\equiv 0$, but since $v = y'$ and $y'(0) = -1/2$, $v \equiv 0$ is impossible!]

$$\Rightarrow \int 1\,dv = \int \frac{1}{y^2}\,dy$$

$$\Rightarrow v = \frac{-1}{y} + C$$

We now find C using our initial conditions:

When $x = 0$, we have $y = 2$ and $v = y' = \frac{-1}{2}$, hence

$$v = \frac{-1}{y} + C \Rightarrow \frac{-1}{2} = \frac{-1}{2} + C \Rightarrow C = 0.$$

Thus, $v = \frac{-1}{y}$.

Step 3: Rewrite $v = y'$ as dy/dx and solve the resulting DE for y.

In our example:

$$v = \frac{-1}{y} \Rightarrow \frac{dy}{dx} = \frac{-1}{y}$$ (separable!)

$$\Rightarrow y\,dy = -dx$$

$$\Rightarrow \frac{y^2}{2} = -x + D$$

We solve for D using $y(0) = 2$ once again:

$$\frac{y^2}{2} = -x + D = \frac{2^2}{2} = -0 + D \Rightarrow \underline{D = 2}.$$

Thus, $\frac{y^2}{2} = -x + 2 \Rightarrow y^2 = 4 - 2x$

$$\Rightarrow y = \pm\sqrt{4 - 2x}$$

However, only $\boxed{y = \sqrt{4-2x}}$ satisfies $y(0) = 2$!

Ex: Solve $y'' = e^y \cdot y'$ given $y(3) = 0$, $y'(3) = 1$

Solution: Since the DE involves only y, y', and y'', we let $y' = v$ and $y'' = v \frac{dv}{dy}$. Then

$$y'' = e^y \cdot y' \Rightarrow v \frac{dv}{dy} = e^y \cdot v$$

$$\Rightarrow \frac{dv}{dy} = e^y \qquad \text{(Again, } v \neq 0 \text{ since } v = y' \text{ and } y'(3) = 1.)$$

$$\Rightarrow \int 1 \, dv = \int e^y \, dy$$

$$\Rightarrow v = e^y + C$$

We are given that $y = 0$ and $v = y' = 1$ when $x = 3$, hence $v = e^y + C \Rightarrow 1 = e^0 + C \Rightarrow \underline{C = 0}$.

Thus,

$$v = e^y \Rightarrow \frac{dy}{dx} = e^y \qquad \text{(Separable!)}$$

$$\Rightarrow \int e^{-y} \, dy = \int 1 \, dx$$

$$\Rightarrow -e^{-y} = x + D$$

Using $y(3)=0$ once again, we have

$$-e^{-y} = x + D \Rightarrow -e^{0} = 3 + D \Rightarrow \underline{D = -4}$$

Thus, $-e^{-y} = x - 4 \Rightarrow e^{-y} = 4 - x$

$$\Rightarrow \boxed{y = -\ln(4-x)}$$

<u>Ex:</u> Find the general solution for $y'' - y' = 0$.

<u>Solution:</u> Using the approach for DEs involving just y, y', and y'', we let $y' = v$ and $y'' = v\frac{dv}{dy}$:

$$y'' - y' = 0 \Rightarrow v\frac{dv}{dy} = v$$

$$\Rightarrow \underline{v \equiv 0} \text{ or } \underline{\frac{dv}{dy} = 1}$$

If $v \equiv 0$, then y = constant and the DE becomes $0 = 0$ (which is true!), so $\underline{y \equiv C \text{ is a possibility.}}$

If instead $v \not\equiv 0$, then

$$\frac{dv}{dy} = 1 \Rightarrow \int 1\, dv = \int 1\, dy$$

$$\Rightarrow v = y + D$$

$$\Rightarrow \frac{dy}{dx} = y + D$$

$$\Rightarrow \int \frac{dy}{y+D} = \int 1\, dx$$

What if $y + D \equiv 0$?

Not possible. Note that $v = y + D$ and we've assumed in this case that $v \not\equiv 0$!

$$\Rightarrow \ln|y+D| = x + E$$

$$\Rightarrow |y+D| = e^x e^E$$

$$\Rightarrow y + D = \pm e^E e^x$$

$$\Rightarrow y = -D \pm e^E \cdot e^x$$

$$\Rightarrow \underline{y = C_1 + C_2 e^x \quad (C_2 = \pm e^E, C_2 \neq 0)}$$

Combining the results from the $v \equiv 0$ case and $v \not\equiv 0$ case, our general solution is

$$\boxed{y = C_1 + C_2 e^x, \; C_1 \in \mathbb{R}, \; C_2 \neq 0 \quad \underline{\underline{\text{or}}} \quad y = C_3, \; C_3 \in \mathbb{R}}$$

Alternatively,

$$\boxed{y = C_1 + C_2 e^x, \; C_1, C_2 \in \mathbb{R}}$$

We get the constant solutions by allowing $C_2 = 0$!

Note: The DE in the last example can also be solved using the method from Case I, as y is not present. Try this as an exercise!