§10.8 (continued) – Recursive Sequences

In the last section we learned that many *limit* techniques from MATH 116 (e.g., squeeze theorem, L'Hopital) can be used to find limits of sequences in MATH 118. These methods work well for explicit sequences (e.g., $\left\{\frac{n}{2n+1}\right\}$) but are difficult to apply to sequences defined recursively.

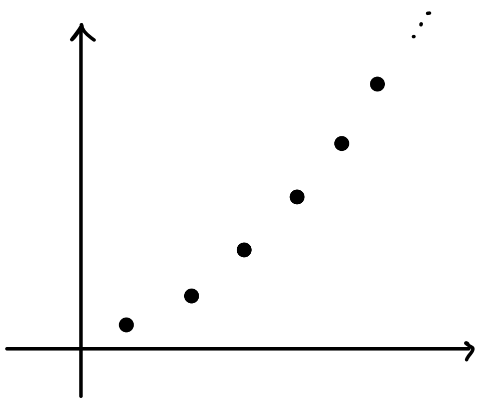
To discuss convergence of recursive sequences, we'll need some new terminology.

Definition: A sequence $\{a_n\}$ is said to be

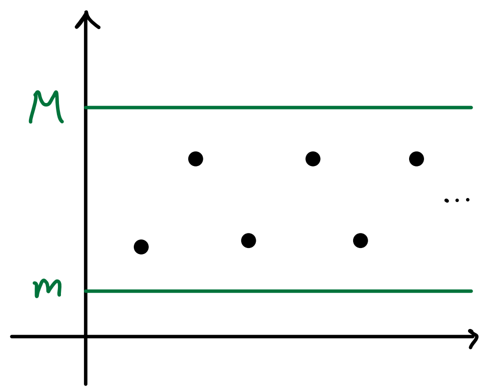
- <u>increasing</u> if $a_n \leq a_{n+1}$ for all n.
- <u>decreasing</u> if $a_n \geq a_{n+1}$ for all n
- <u>monotone</u> (or <u>monotonic</u>) if $\{a_n\}$ is either increasing or decreasing (or both, in the case of a constant sequence).

- bounded if there exist numbers m and M such that $m \le a_n \le M$ for all n.

Examples (by picture!):

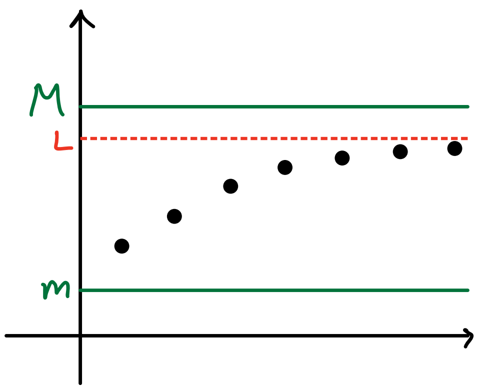


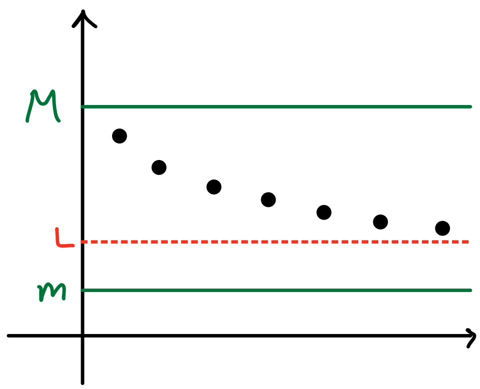
Increasing (hence monotone) but not bounded



Bounded, but not monotone.

But observe that if a sequence is both monotone and bounded...





... then it must "level off" at some value L!

That is, the sequence must converge! This gives us the following theorem.

The Monotone Sequence Theorem (MST)

Every bounded monotone sequence converges.

Note: $\{a_n\}$ may not converge to m or M, it could converge to some other number L in between!

The MST will be our main technique for proving that certain recursive sequences converge: we can try to show that they are bounded and monotone!

Example: Let's consider the recursive sequence

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_{n+1} = \frac{1+a_n}{2}, \quad n \geq 1$$

Let's write out a few terms to see what's going on...

$$a_1 = \frac{1}{2}$$

$$a_2 = \frac{1+a_1}{2} = \frac{1+1/2}{2} = \frac{3}{4}$$

$$a_3 = \frac{1+a_2}{2} = \frac{1+3/4}{2} = \frac{7}{8}$$

$$a_4 = \frac{1+a_3}{2} = \frac{1+7/8}{2} = \frac{15}{16}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Based on these terms, we guess that

- $\{a_n\}$ is increasing (hence monotone)
- $\{a_n\}$ is bounded with $\frac{1}{2} \le a_n \le 1$.

But how can we know for sure??

To prove that $\{a_n\}$ is increasing, for example, we would need to prove the following infinitely many propositions:

"Proposition 1"

$P_1: \quad a_1 \leq a_2$

$P_2: \quad a_2 \leq a_3$

$P_3: \quad a_3 \leq a_4$

$\vdots \qquad \vdots$

$P_n: \quad a_n \leq a_{n+1}$

$\vdots \qquad \vdots$

We can do this using a technique known as ...

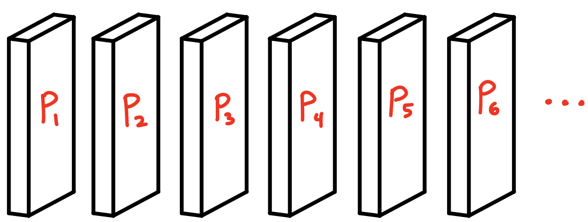
Mathematical Induction

Step 1: Prove that P_1 is true. (The base case)

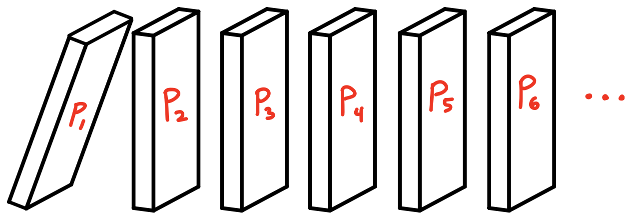
Step 2: Prove that if one of the statements P_n is true, then the next statement P_{n+1} must also be true.

(The inductive step)

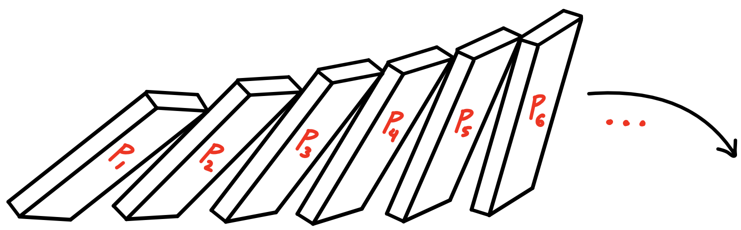
Think of each statement as a domino and proving the statement as knocking the domino over.



If we can knock down the first domino (i.e., prove P_1)...



AND we can show that when any domino falls it must also knock down the domino next to it (i.e., prove that if P_n is true, then so is P_{n+1}) ...



then this will be enough to knock down the whole stack (i.e., prove all the statements!)

<u>Ex:</u> Consider the sequence $a_1 = \frac{1}{2}$, $a_{n+1} = \frac{1+a_n}{2}$ $(n \geq 1)$.

Prove that a_n is increasing (i.e., $a_n \leq a_{n+1}$ for all n).

Proof (by induction!):

We will prove the statements $P_n : a_n \le a_{n+1}$ $(n \in \mathbb{N})$

Base Case:

We first prove P_1. That is, we prove $a_1 \le a_2$. Since

$a_1 = 1/2$ and $a_2 = \frac{1+a_1}{2} = 3/4$, clearly $a_1 \le a_2$, as needed.

Inductive Step:

Assume now that P_n is true for <u>some</u> $n \in \mathbb{N}$. That is

assume $a_n \le a_{n+1}$. We will prove P_{n+1}, that $a_{n+1} \le a_{n+2}$.

Indeed,

$$a_n \le a_{n+1} \Rightarrow 1 + a_n \le 1 + a_{n+1}$$

$$\Rightarrow \underbrace{\frac{1+a_n}{2}}_{=a_{n+1}} \le \underbrace{\frac{1+a_{n+1}}{2}}_{=a_{n+2}}$$

$$\Rightarrow a_{n+1} \le a_{n+2}, \text{ as desired!}$$

Therefore, by induction $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$, hence

$\{a_n\}$ is increasing. ■

<u>Ex:</u> Once again, consider the sequence

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_{n+1} = \frac{1+a_n}{2}, \quad n \in \mathbb{N}$$

Prove that $\frac{1}{2} \le a_n \le 1$ for all n.

<u>Proof (by induction!):</u>

We will prove the statements $P_n : \frac{1}{2} \le a_n \le 1 \quad (n \in \mathbb{N})$

<u>Base Case:</u>

We first prove P_1. Since $a_1 = \frac{1}{2}$, it is clear that $\frac{1}{2} \le a_1 \le 1$, so P_1 is true.

<u>Inductive Step:</u>

Assume now that P_n is true for some $n \in \mathbb{N}$. We will prove P_{n+1}, that $\frac{1}{2} \le a_{n+1} \le 1$. We have

$$\frac{1}{2} \le a_n \le 1 \Rightarrow 1 + \frac{1}{2} \le 1 + a_n \le 1 + 1$$

$$\Rightarrow \underbrace{\frac{1+\frac{1}{2}}{2}}_{=3/4} \le \underbrace{\frac{1+a_n}{2}}_{=a_{n+1}} \le \underbrace{\frac{1+1}{2}}_{=1}$$

$$\Rightarrow \quad 3/4 \le a_{n+1} \le 1$$

Since $\frac{1}{2} \leq \frac{3}{4}$, we have $\frac{1}{2} \leq a_{n+1} \leq 1$, as required!

By induction, $a_n \leq 1$ for all $n \in \mathbb{N}$. ∎

We have now shown that the sequence

$$a_1 = \tfrac{1}{2}, \quad a_{n+1} = \frac{1+a_n}{2} \quad (n \in \mathbb{N})$$

is increasing (hence monotone) and bounded. Thus, by the Monotone Sequence Theorem, $\{a_n\}$ converges!

Once we know $L = \lim_{n\to\infty} a_n$ exists, we do the following:

$$a_{n+1} = \frac{1+a_n}{2} \implies \lim_{n\to\infty} \underbrace{a_{n+1}}_{\to L} = \lim_{n\to\infty} \frac{1+\overbrace{a_n}^{\to L}}{2}$$

$$\implies L = \frac{1+L}{2}$$

$$\implies 2L = 1 + L$$

$$\implies \underline{L = 1}$$

Hence, $\boxed{\lim_{n\to\infty} a_n = 1.}$

Ex: Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_1 = 2 \quad \text{and} \quad a_{n+1} = \frac{1}{3-a_n}, \quad n \geq 1.$$

Prove that the sequence converges and find its limit.

Solution: Let's write out some terms to see what's going on here:

$$\boxed{\begin{aligned} a_1 &= \underline{2} \\ a_2 &= \frac{1}{3-a_1} = \frac{1}{3-2} = \underline{1} \\ a_3 &= \frac{1}{3-a_2} = \frac{1}{3-1} = \frac{1}{2} = \underline{0.5} \\ a_4 &= \frac{1}{3-a_3} = \frac{1}{3-\frac{1}{2}} = \frac{2}{5} = \underline{0.4} \\ &\vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}}$$

We therefore guess that

(i) $\underline{0 < a_n \leq 2 \text{ for all } n}$, and

(ii) $\underline{\{a_n\} \text{ is decreasing}}$.

We will prove these statements separately.

(i) Proof that $0 < a_n \leq 2$ for all n

P_n is the statement that $0 < a_n \leq 2 \quad (n \geq 1)$

Base Case:

We have $a_1 = 2$, hence $0 < a_1 \leq 2$, so P_1 is true.

Inductive Step:

Assume that P_n is true for some n, so $0 < a_n \leq 2$.

We will prove P_{n+1}, that $0 < a_{n+1} < 2$. Indeed,

$$
\begin{aligned}
0 < a_n \leq 2 \quad &\Rightarrow \quad 0 > -a_n \geq -2 && \text{(multiply by } -1\text{)} \\
&\Rightarrow \quad 3 > 3 - a_n \geq 1 && \text{(add 3)} \\
&\Rightarrow \quad \frac{1}{3} < \underbrace{\frac{1}{3-a_n}}_{=a_{n+1}} \leq 1 && \text{(reciprocate)} \\
&\Rightarrow \quad \frac{1}{3} < a_{n+1} \leq 1
\end{aligned}
$$

Consequently, $0 < a_{n+1} \leq 2$, so P_{n+1} is true.

By induction, $0 < a_n \leq 2$ for all n. $\blacksquare$

(ii) Proof that $\{a_n\}$ is decreasing

P_n is the statement that $a_n \geq a_{n+1}$ $(n \geq 1)$

Base Case:

We will prove P_1, that $a_1 \geq a_2$. Indeed, $a_1 = 2$ and $a_1 = 1$, hence $a_1 \geq a_2$.

Inductive Step:

Assume that P_n is true for some n, so $a_n \geq a_{n+1}$.

We will prove P_{n+1}, that $a_{n+1} \geq a_{n+2}$. Indeed,

$$a_n \geq a_{n+1} \Rightarrow -a_n \leq -a_{n+1} \quad \text{(multiply by } -1\text{)}$$

$$\Rightarrow 3 - a_n \leq 3 - a_{n+1} \quad \text{(add 3)}$$

$$\Rightarrow \frac{1}{3-a_n} \geq \frac{1}{3-a_{n+1}} \quad \text{(reciprocate)}$$

$$\Rightarrow a_{n+1} \geq a_{n+2}, \quad \text{as desired.}$$

$\therefore$ By induction, $a_n \geq a_{n+1}$ for all n, hence $\{a_n\}$ is decreasing. $\blacksquare$

Okay! We've successfully shown that $\{a_n\}$ is both bounded and monotonic, hence $\{a_n\}$ converges to a limit L by the Monotone Sequence Theorem.

<u>What is L?</u>

$$a_{n+1} = \frac{1}{3-a_n} \implies \lim_{n\to\infty} \underbrace{a_n}_{\to L} = \lim_{n\to\infty} \frac{1}{3-\underbrace{a_n}_{\to L}}$$

$$\implies L = \frac{1}{3-L}$$

$$\implies 3L - L^2 = 1$$

$$\implies L^2 - 3L + 1 = 0$$

$$\implies L = \frac{3 \pm \sqrt{5}}{2}$$

which one is our limit?

Since $0 < a_n \leq 2$, we must have $0 \leq L \leq 2$.

However, $\frac{3+\sqrt{5}}{2} \approx 2.618$ (too big!). Hence

$$\boxed{\lim_{n\to\infty} a_n = \frac{3-\sqrt{5}}{2} \approx 0.382}$$

Caution: Don't solve for L until you have proved that L exists (using the MST), otherwise it can lead to incorrect conclusions!

e.g. The sequence $a_1 = 2$ and $a_{n+1} = \frac{1}{a_n}$ $(n \geq 1)$ is given by $a_1 = 2$, $a_2 = \frac{1}{2}$, $a_3 = 2$, $a_4 = \frac{1}{2}$, ... and hence is divergent. However, if we (incorrectly) supposed that $a_n \longrightarrow L$ as $n \longrightarrow \infty$, we would find that

$$a_{n+1} = \frac{1}{a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{a_n}$$

$$\Rightarrow L = \frac{1}{L}$$

$$\Rightarrow L^2 = 1$$

$$\Rightarrow L = \pm 1$$

Both of these are wrong, the sequence doesn't have a limit!

Additional Exercises:

1. Consider the sequence

$$a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{1+a_n}{2} \quad (n \in \mathbb{N})$$

By writing out the first several terms

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{3}{4}, \quad a_3 = \frac{7}{8}, \quad a_4 = \frac{15}{16}, \ \dots$$

We might be able to guess an explicit form for $\{a_n\}$:

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \quad (n \in \mathbb{N})$$

(a) Prove using induction that $a_n = 1 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

(b) Use (a) to calculate $\lim_{n\to\infty} a_n$.

Solution:

(a) Proof: We will prove the statements

$$P_n : \ a_n = 1 - \frac{1}{2^n}, \quad n \in \mathbb{N}.$$

Base Case:

We first prove P_1. We have $a_1 = \frac{1}{2}$ and $1 - \frac{1}{2^1} = \frac{1}{2}$, hence $a_1 = 1 - \frac{1}{2^1}$, as claimed.

Inductive Step:

Assume now that P_n is true for some $n \in \mathbb{N}$, so

$a_n = 1 - \frac{1}{2^n}$. We will prove P_{n+1}, that $a_{n+1} = 1 - \frac{1}{2^{n+1}}$.

Indeed,

$$a_n = 1 - \frac{1}{2^n} \implies 1 + a_n = 1 + \left(1 - \frac{1}{2^n}\right)$$

$$\implies \underbrace{\frac{1+a_n}{2}}_{=a_{n+1}} = \underbrace{\frac{2 - \frac{1}{2^n}}{2}}_{= 1 - \frac{1}{2^{n+1}}}$$

$$\implies a_{n+1} = 1 - \frac{1}{2^{n+1}}$$

Thus, by induction, $a_n = 1 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$. ■

(b) It's now easy to determine the behavior of $\{a_n\}$ as $n \to \infty$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \underbrace{\frac{1}{2^n}}_{\to 0}\right) = 1 - 0 = \boxed{1}$$