Rearrangements [Not Tested!]

One important reason to distinguish between absolute and conditional convergence relates to rearrangements: adding the terms of a series in a different order.

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

vs.

$$a_1 + a_3 + a_2 + a_5 + a_4 + \cdots$$

It turns out that rearranging the terms of a conditionally convergent series can actually change its sum!

Ex: Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, which we know converges conditionally. Let S be the sum:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

Let's rearrange the terms!

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

Rearrange

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right] = \frac{1}{2}S \;!!$$

Our rearrangement changed the value of the sum!

The following (AMAZING!) theorem of Riemann shows that this could happen for any conditionally convergent series.

Riemann's Rearrangement Theorem (1852)

If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then one can rearrange the terms to produce <u>any</u> sum (or $\pm\infty$).

This means there exist rearrangements of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ that sum to π, e, $-\sqrt{7}$, $\pi^{e^{\pi^2}}$ and any other number you can think of!

Thankfully, not all infinite series behave this way!

> <u>Fact</u>: If $\sum a_n$ <u>converges absolutely</u> with sum S, then any rearrangement will also sum to S.