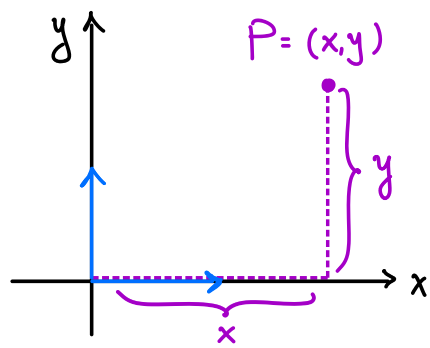
# §9.2 – Polar Coordinates

We're used to describing the location of a point in space using its $xy$-coordinates (a.k.a., Cartesian coordinates)

$x$ = how far left/right from $(0,0)$.

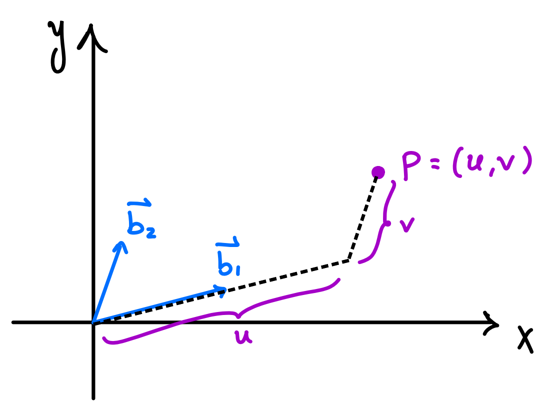
$y$ = how far up/down from $(0,0)$.




In MATH 115, you learned that the location of $P$ can be described in other coordinate systems too (i.e., relative to a different basis)

$u$ = how far to move in the direction of $\vec{b}_1$.

$v$ = how far to move in the direction of $\vec{b}_2$.

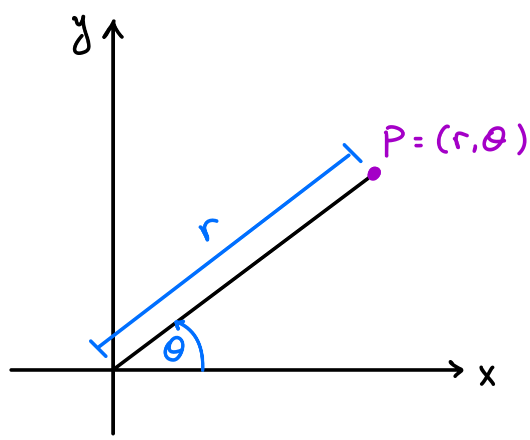


Alternatively, we can specify the location of P by reporting

$\theta$ : the angle P makes with the positive x-axis (measured counterclockwise),

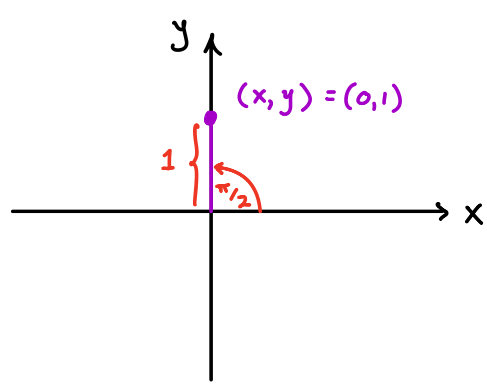
$r$ : the distance from P to the origin.

We call $(r, \theta)$ the polar coordinates of P.

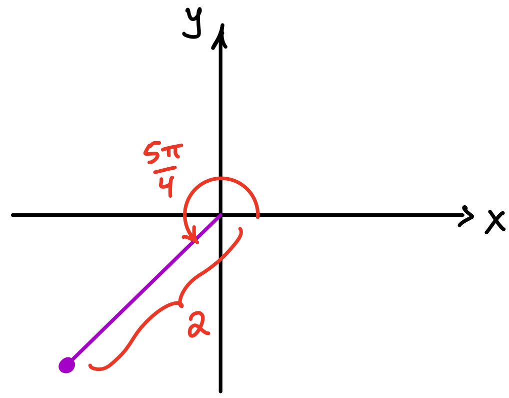



Let's plot some points given their polar coordinates!

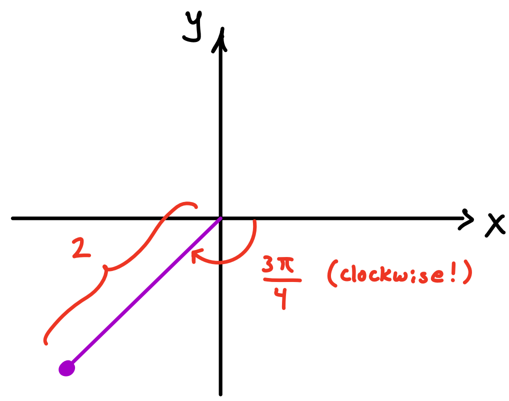
(a) $(r, \theta) = (1, \pi/2)$



(b) $(r,\theta) = \left(2, \frac{5\pi}{4}\right)$




(c) $(r,\theta) = \left(2, -\frac{3\pi}{4}\right)$




Notice that $(r,\theta) = \left(2, \frac{5\pi}{4}\right)$ and $(r,\theta) = \left(2, -\frac{3\pi}{4}\right)$ actually describe the same point P. In fact,

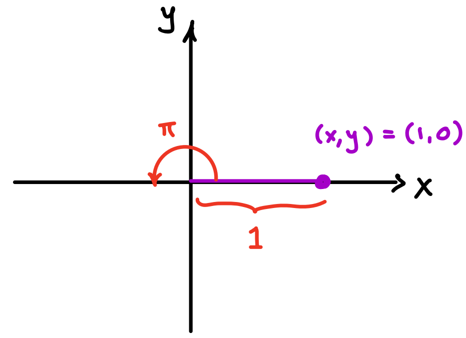
$$\left(2, \frac{5\pi}{4} + 2k\pi\right), \quad k \in \mathbb{Z}$$

all describe this same point too!

$$\boxed{\text{A point P has infinitely many representations in polar coordinates!}}$$

(d) $(r,\theta) = (-1, \pi)$

Negative $r$ means go in opposite direction

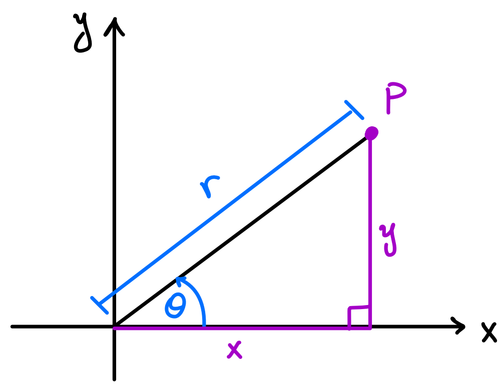



To convert between Cartesian and polar coordinates, we can use the following formulas:

Given $(r,\theta)$, we have

$$\frac{x}{r} = \cos\theta \Rightarrow \quad x = r\cos\theta,$$

$$\frac{y}{r} = \sin\theta \Rightarrow \quad y = r\sin\theta$$




Ex: Find the Cartesian coordinates of $(r,\theta) = (2, \frac{5\pi}{4})$.

Solution: $x = r\cos\theta = 2\cos\frac{5\pi}{4}$

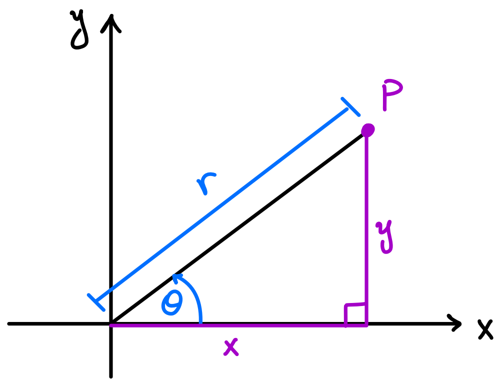
$= 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

$y = r\sin\theta = 2\sin\frac{5\pi}{4}$

$$= 2\left(\frac{-\sqrt{2}}{2}\right) = -\sqrt{2}$$

Thus, $\boxed{(x, y) = (-\sqrt{2}, -\sqrt{2}).}$

Given $(x,y)$, we have




$$\boxed{r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2+y^2} \quad \text{(if we take } r \geq 0\text{)}}$$

$$\boxed{\frac{x}{r} = \cos\theta \;, \quad \frac{y}{r} = \sin\theta}$$

Solve for $\theta$!

Ex: Find polar coordinates for $(x,y) = (-1, \sqrt{3})$.

Solution: $r = \sqrt{x^2+y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

and since
$$\left.\begin{array}{l} \cos\theta = \frac{x}{r} = \frac{-1}{2} \\ \sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2} \end{array}\right\} \quad \theta = \frac{2\pi}{3} \quad (+2k\pi,\ k \in \mathbb{Z})$$

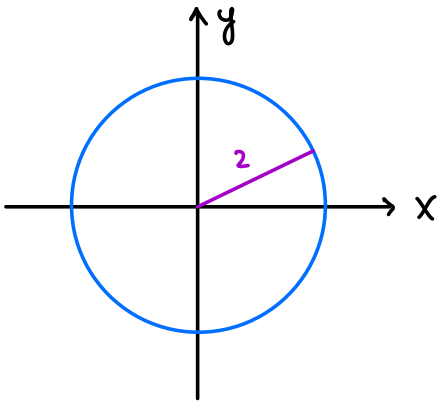
One solution is $\boxed{(r, \theta) = \left(2, \frac{2\pi}{3}\right).}$

## Polar Curves

Let's sketch some curves defined in terms of $r$ & $\theta$!

**Ex:** $r = 2$

**Solution:** All points that are 2 units from the origin



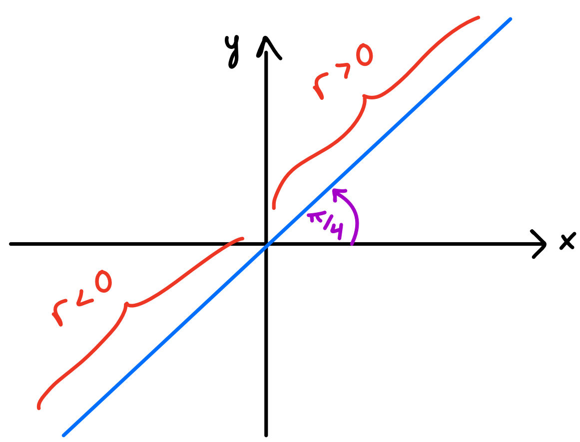

We can also see this by converting to Cartesian:

$$r = 2 \Rightarrow \sqrt{x^2+y^2} = 2 \Rightarrow x^2+y^2 = 4$$

(circle of radius 2)

**Ex:** $\theta = \pi/4$

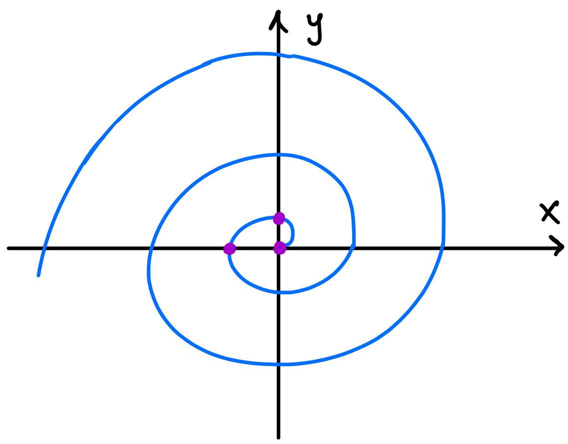
**Solution:** All points that make an angle of $\pi/4$ with the positive x-axis



Ex: $r = \theta$, $\theta \geq 0$.

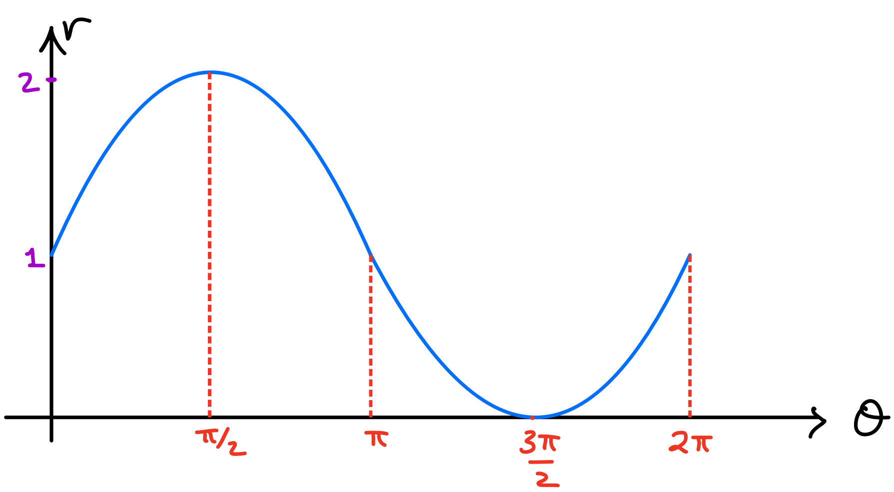
Solution: This curve passes through $(r,\theta) = (0,0)$,

$(r,\theta) = (\pi/2, \pi/2)$, $(r,\theta) = (\pi,\pi)$, etc...



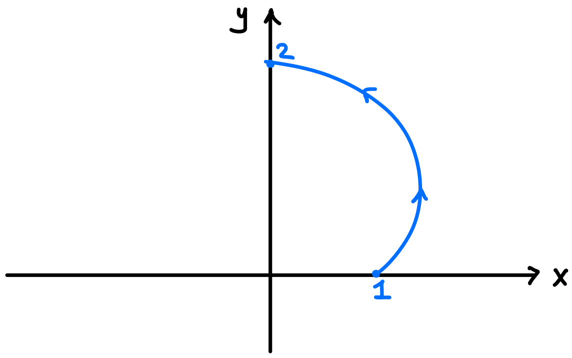

Ex: $r = 1 + \sin\theta$

Solution: For more complicated curves like this, start by sketching the graph in the $r\theta$-plane

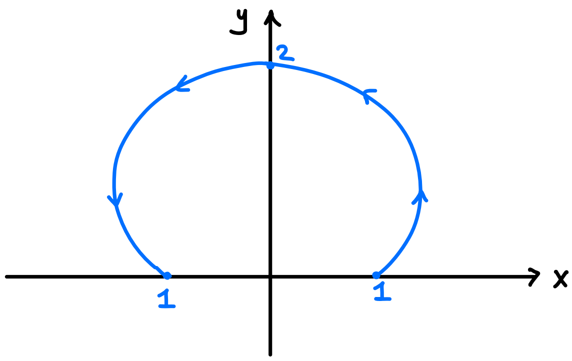


Now use this to sketch the graph in the xy-plane.

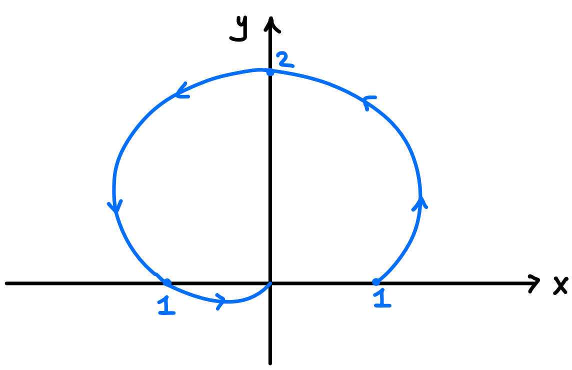
$\underline{\theta \in [0, \pi/2]}$: $r$ increases from $r=1$ to $r=2$.



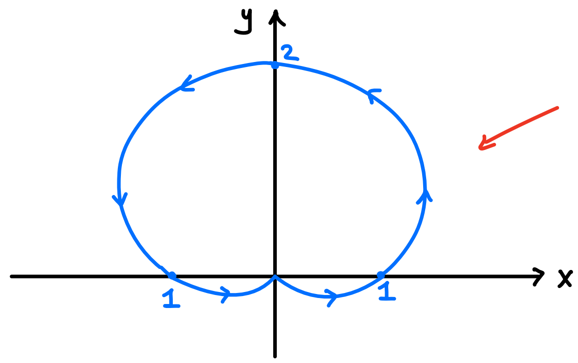

$\underline{\theta \in [\pi/2, \pi]}$: $r$ decreases from $r=2$ to $r=1$.




$\underline{\theta \in [\pi, \frac{3\pi}{2}]}$: $r$ decreases from $r=1$ to $r=0$.



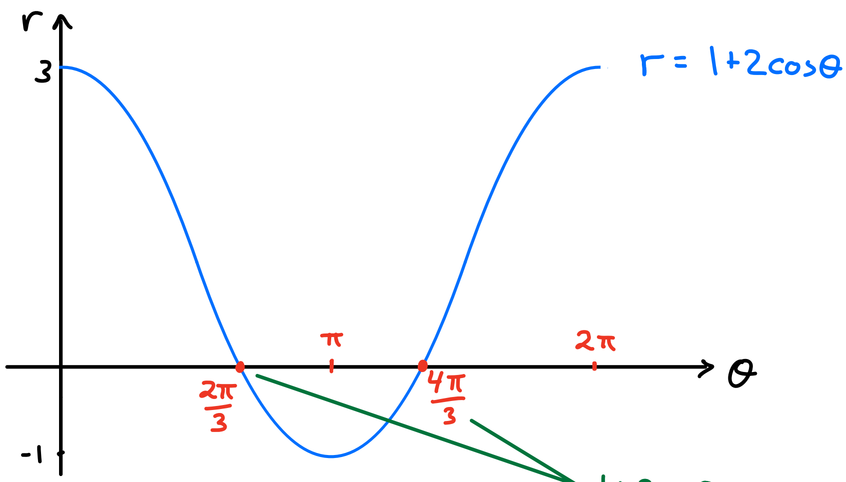
$\theta \in [\frac{3\pi}{2}, 2\pi]$: r increases from $r=0$ to $r=1$.




This curve is called a cardioid

Ex: Sketch the graph of $r = 1 + 2\cos\theta$.

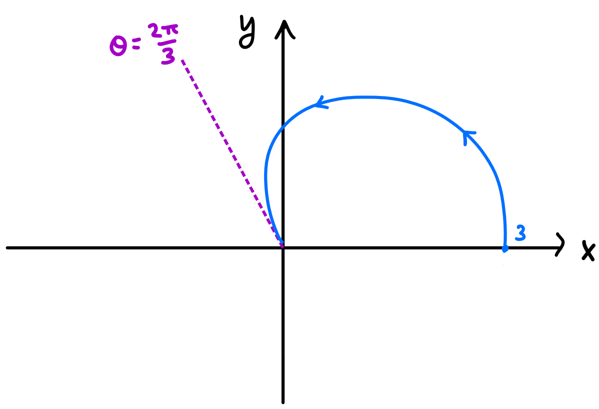
Solution:




$1 + 2\cos\theta = 0 \Rightarrow \cos\theta = \frac{-1}{2}$

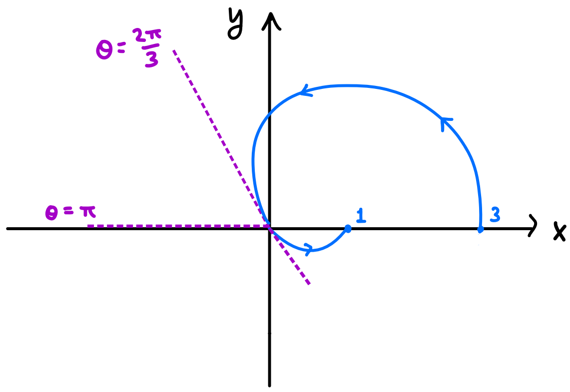
$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\theta \in [0, \frac{2\pi}{3}]$: r decreases from $r=3$ to $r=0$.

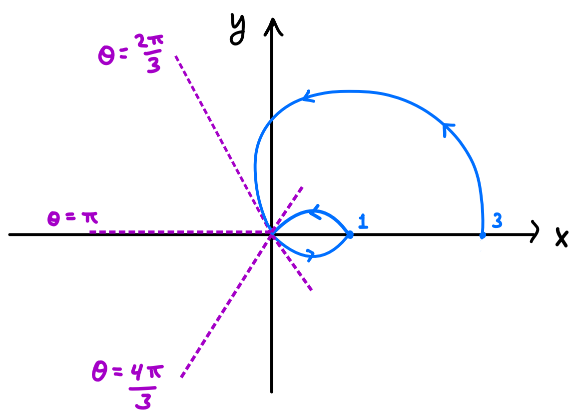


$\underline{\theta \in \left[\frac{2\pi}{3}, \pi\right]}$: r decreases from 0 to -1

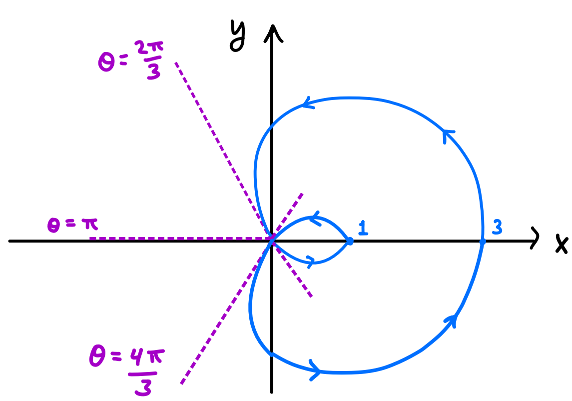
(or, r increases from 0 to 1 but in the opposite direction!)




$\underline{\theta \in \left[\pi, \frac{4\pi}{3}\right]}$: r increases from -1 to 0.

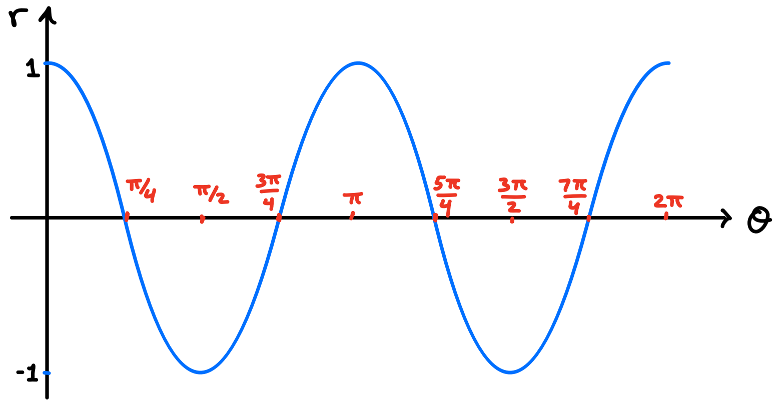



$\underline{\theta \in \left[\frac{4\pi}{3}, 2\pi\right]}$: r increases from 0 to 3.

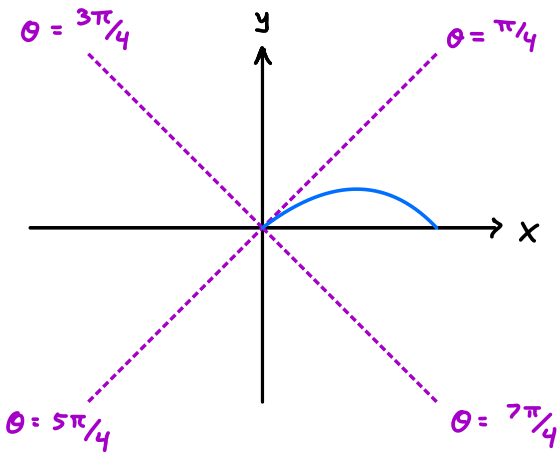


**Ex:** Sketch the graph of $r = \cos(2\theta)$ for $\theta \in [0, 2\pi]$.

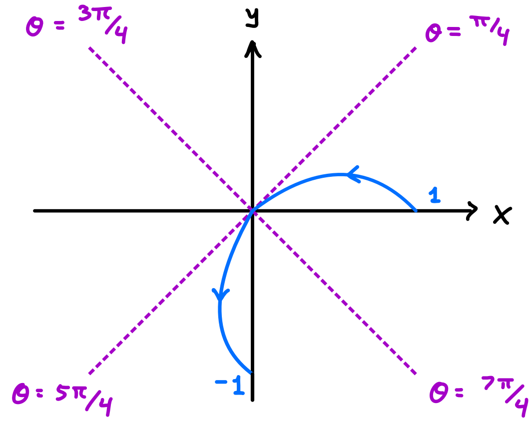
**Solution:**



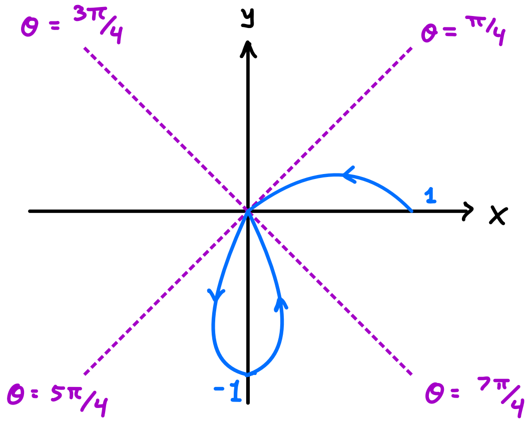

$\theta \in [0, \pi/4]$: $r$ decreases from 1 to 0



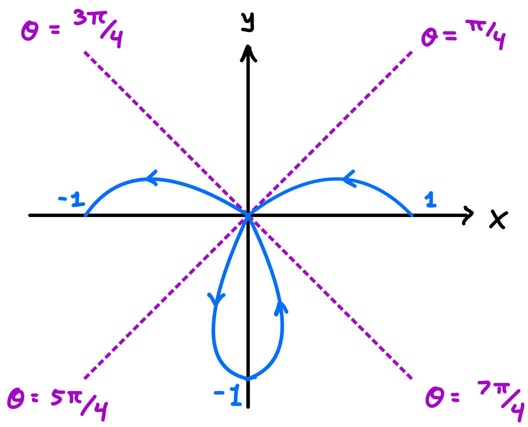

$\theta \in [\pi/4, \pi/2]$: $r$ decreases from 0 to $-1$.



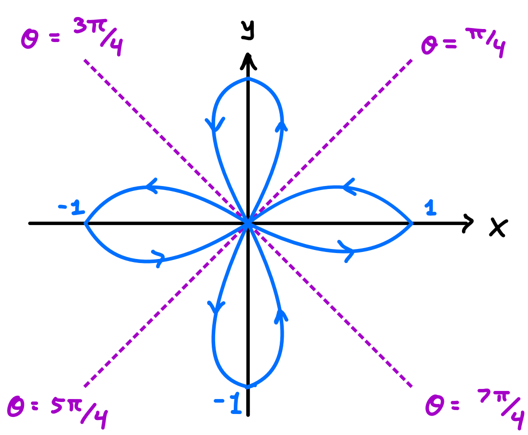
$\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$: $r$ increases from $-1$ to $0$.




$\theta \in [\frac{3\pi}{4}, \pi]$: $r$ increases from $0$ to $1$.




Similar behaviour occurs on the remaining intervals:



**Ex:** Sketch the graph of $r = 2\cos\theta$.

**Solution:** In Cartesian coordinates, we have

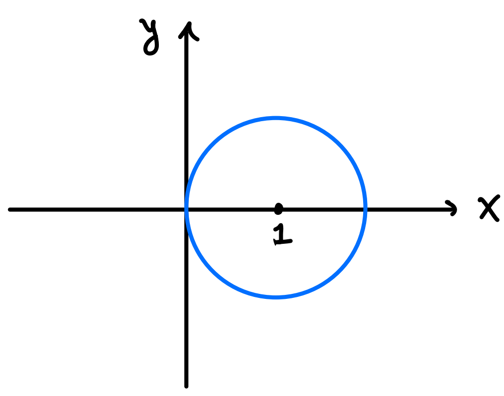
$$r = 2\cos\theta \Rightarrow \sqrt{x^2+y^2} = 2\left(\frac{x}{r}\right) = 2\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$\Rightarrow x^2+y^2 = 2x$$

$$\Rightarrow x^2-2x+y^2 = 0$$

$$\Rightarrow (x-1)^2+y^2 = 1$$ (complete the square!)

So we're expecting a circle centred at $(1,0)$ of radius 1.




[See if you can arrive at the same curve without converting to Cartesian coordinates. Instead, use the same approach as in our earlier examples.]