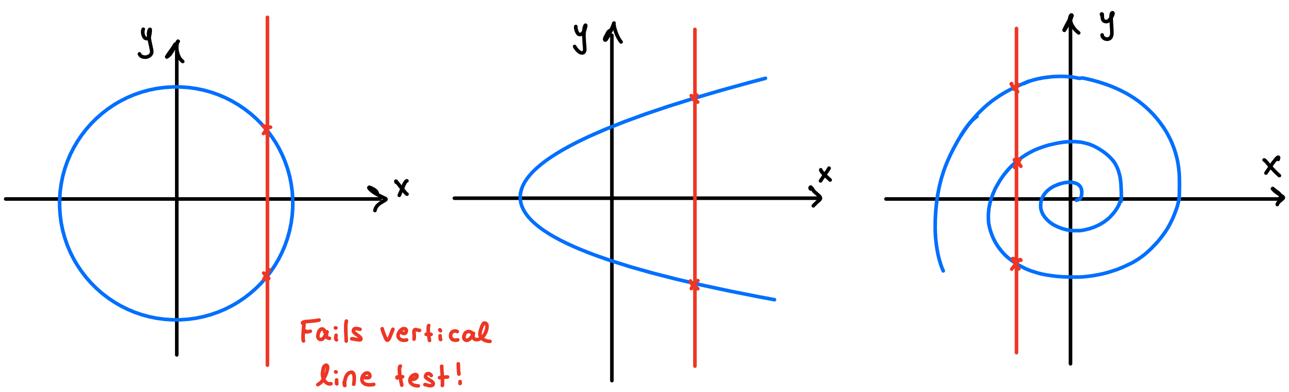
§ 9.1 - Parametric Curves

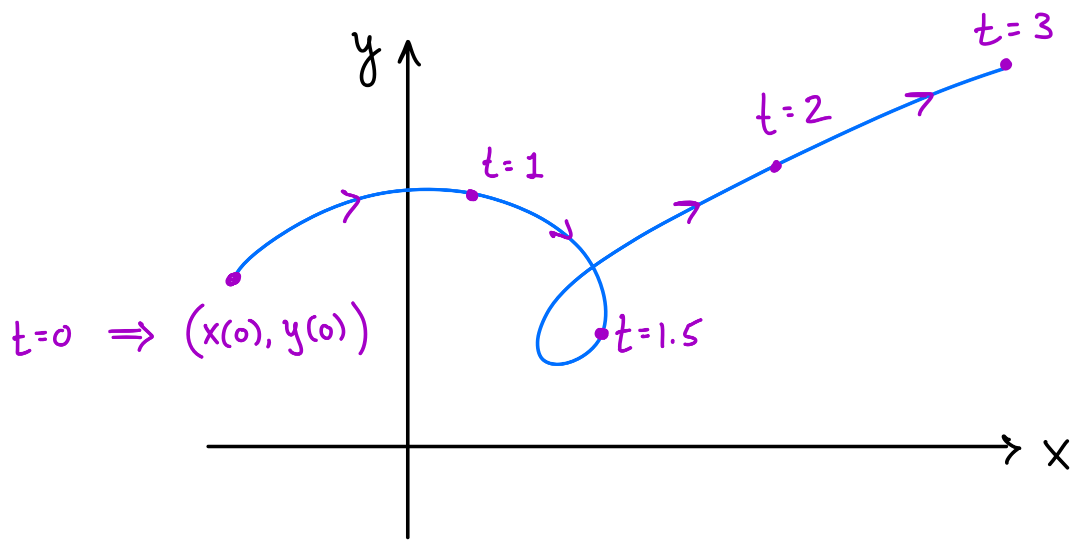
Up to now, we have mainly been working with curves defined by $y = f(x)$; that is, graphs of functions. But there are many interesting curves that are not defined by a function $y = f(x)$:



These curves can instead be described using parametric equations, where both x and y are functions of a parameter, t:

$$\boxed{x = f(t), \quad y = g(t), \quad t \in [a, b].}$$

(often, this is written $x = x(t)$, $y = y(t)$, $t \in [a,b]$.)



You can think of $x(t)$ and $y(t)$ as the (x,y) coordinates of a bug at time t. The curve traced out as t increases from $t=a$ to $t=b$ is called a parametric curve.

Ex: Consider

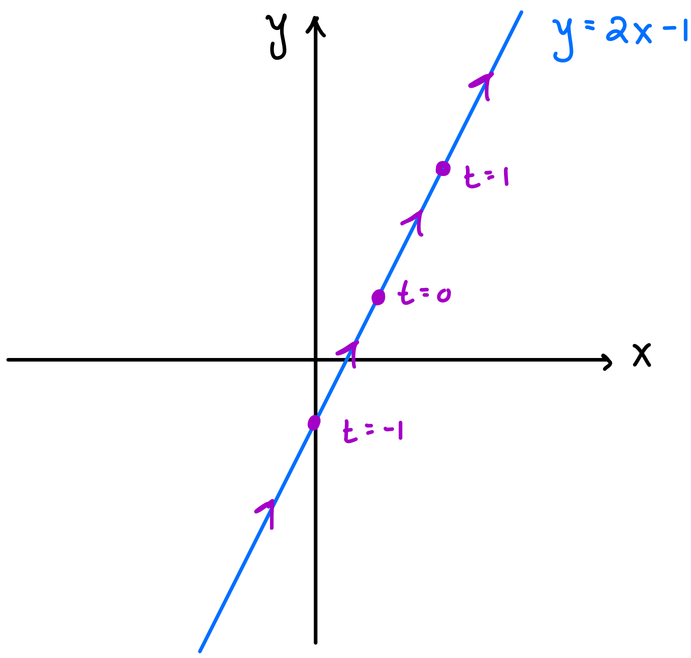
$$x(t) = t+1$$
$$y(t) = 2t+1 \qquad \text{for } t \in (-\infty, \infty)$$

which you may recognize from MATH 115 as the parametric equation of a line!

We can plot the line by eliminating t:

$$x = t+1 \Rightarrow t = x-1,$$

which means $y = 2t+1 = 2(x-1)+1 \Rightarrow \underline{y = 2x-1}$



Note: The parametric equation tells us more than $y = f(x)$ — it specifies the direction in which the curve is traced!

Ex: Plot $x = t^2 - 2t$, $y = t+1$.

(Here, the domain of t is not stated. Assume all possible t's are allowed!)

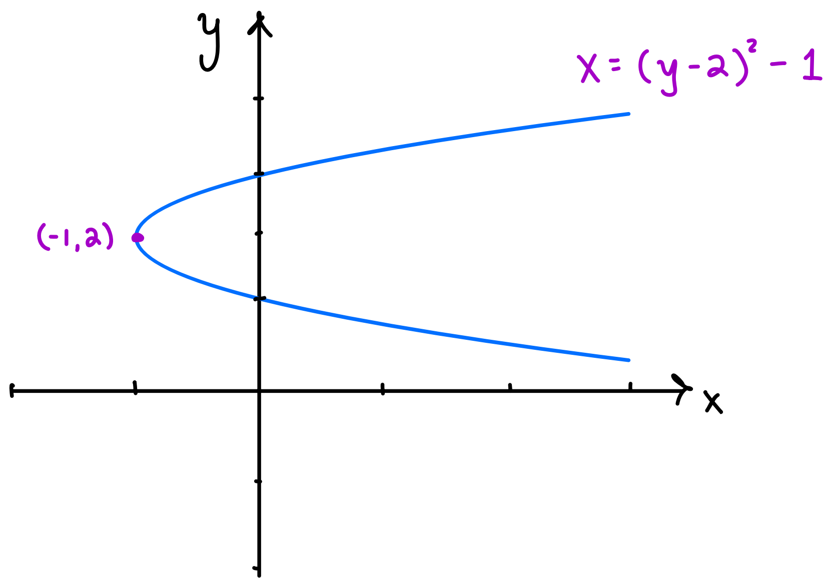
Solution: $y = t+1 \Rightarrow t = y-1$

Thus, $x = t^2 - 2t = (y-1)^2 - 2(y-1)$

$$= (y^2 - 2y + 1) - 2y + 2$$

$$= y^2 - 4y + 3$$

$$= (y-2)^2 - 1$$ (complete the square!)

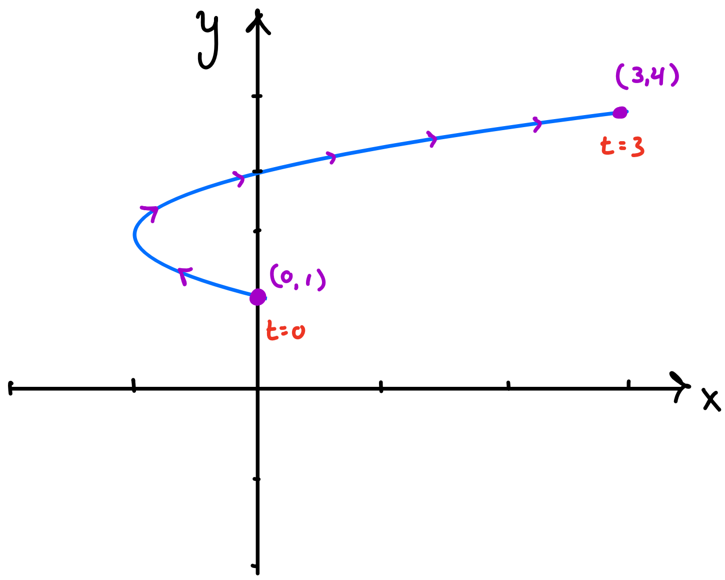


Follow-up: What if we insisted that $t \in [0,3]$?

Answer: We would only see a piece of the parabola!

$t = 0 \Rightarrow (x,y) = (t^2 - 2t, t+1) = (0,1)$

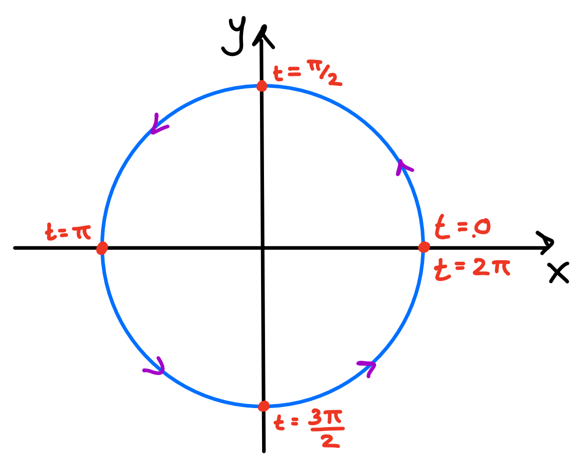
$t = 3 \Rightarrow (x,y) = (t^2 - 2t, t+1) = (3,4)$



<u>Ex:</u> Plot $x = \cos t$, $y = \sin t$ for $t \in [0, 2\pi]$.

<u>Solution:</u> $\cos^2 t + \sin^2 t = 1 \Rightarrow x^2 + y^2 = 1$

We recognize this as the unit circle traversed once counterclockwise.



<u>Follow-up:</u> Does $x = \cos(2t)$, $y = \sin(2t)$, $t \in [0, 2\pi]$ also describe the unit circle?

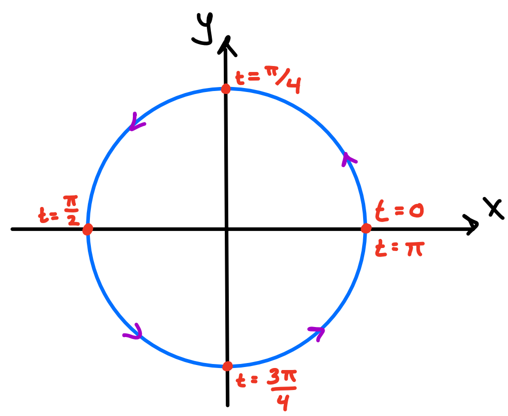
Solution: We still have

$$x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$$

so the equations still describe the unit circle.

This time, however, we complete one revolution for $t \in [0, \pi]$ and another revolution for $t \in [\pi, 2\pi]$.

So the curve looks the same, but now we make two revolutions!



> **The moral:** Different parametric equations can describe the same parametric curve!