§ 8.7 – Numerical Integration

At this point we have learned a variety of integration techniques and can integrate many types of functions. But can we integrate every function? No!

$$e^{x^2}, \quad \sin(x^2), \quad \cos(x^2), \quad \sin(e^x), \quad \frac{1}{\ln x}, \quad \sqrt{x^3+1}$$

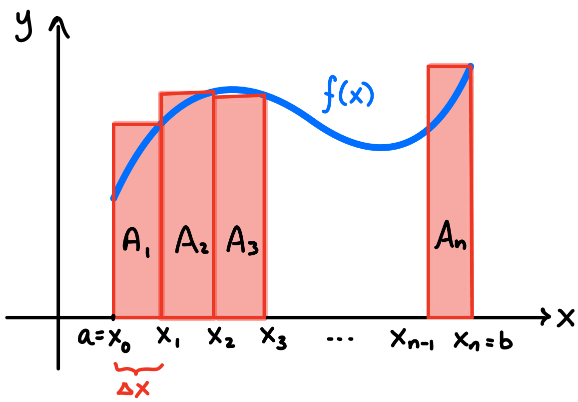
don't have elementary antiderivatives (i.e., antiderivatives expressible as combinations of exponentials, logs, trig functions, polynomials, etc...). Thus,

$$\int_0^1 e^{x^2}\,dx, \quad \int_0^{2\pi} \sin(x^2)\,dx, \quad \text{etc.}$$

cannot be evaluated using the FTC! Our goal is therefore to learn how to approximate integrals that we can't easily evaluate.

1. Rectangular Approximation (MATH 116)

We can approximate the area under a curve using left endpoint or right endpoint rectangles:



Using n subintervals of $[a,b]$, we have $\boxed{\Delta x = \frac{b-a}{n}}$ and

$\boxed{x_i = a + i\Delta x.}$

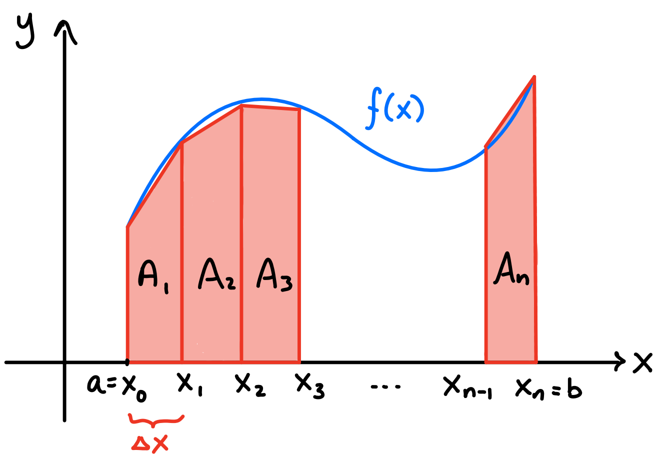
Thus, if we use right endpoints for the height of each rectangle, we get $A_i = f(x_i)\Delta x$, and hence

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx = \text{Area} &\approx A_1 + A_2 + \cdots + A_n \\ &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]\end{aligned}}$$

Right-endpoint Riemann Sum

Let's see now see how we can approximate with trapezoids!

2. Trapezoid Rule



Using n subintervals, we still have $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. Here

$$A_i = \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right]\Delta x$$

Hence...

Trapezoid Rule!

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &\approx A_1 + A_2 + \cdots + A_n \\ &= \left[\frac{f(x_0)+f(x_1)}{2}\right]\Delta x + \left[\frac{f(x_1)+f(x_2)}{2}\right]\Delta x + \cdots \\ &\qquad \cdots + \left[\frac{f(x_{n-1})+f(x_n)}{2}\right]\Delta x \\ &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]\end{aligned}}$$

Ex: Approximate $\int_1^3 \frac{1}{x}\,dx$ using the trapezoid rule with $n=5$ subintervals.

<u>Solution:</u> $n=5$, so $\Delta x = \frac{b-a}{n} = \frac{3-1}{5} = \frac{2}{5} = 0.4$

and $x_i = a + i\Delta x = 1 + 0.4i$ $\quad (= 1, 1.4, 1.8, 2.2, 2.6, 3)$

We have

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right]$$

$$= \frac{0.4}{2}\left[\frac{1}{1} + \frac{2}{1.4} + \frac{2}{1.8} + \frac{2}{2.2} + \frac{2}{2.6} + \frac{1}{3}\right]$$

$$\approx \boxed{1.11027}$$

How close is this to the actual value of the integral?

We define the <u>error</u> in our trapezoid approximation to be

$$T_n = (\text{actual area}) - (\text{approximation})$$

In our example, $\int_1^3 \frac{1}{x}\,dx = \ln 3 - \ln 1 = \ln 3$, and so

$$T_5 \approx \ln 3 - 1.11027 \approx -0.0117$$

<u>Q:</u> How can we understand the size of the error without already knowing the value of $\int_a^b f(x)\,dx$?

Error Bound for the Trapezoid Rule

If we approximate $\int_a^b f(x)\,dx$ using the trapezoid rule with n subintervals, then

$$\boxed{|T_n| \leq \frac{M(b-a)^3}{12n^2},}$$

where M is an upper bound for $|f''(x)|$ for $x \in [a,b]$.

Earlier, we approximated $\int_1^3 \underbrace{\frac{1}{x}}_{=f(x)}\,dx$ using the trapezoid rule with $n=5$. The error satisfies

$$|T_5| \leq \frac{M(3-1)^3}{12 \cdot 5^2} = \frac{8M}{300},$$

where M is the maximum of $|f''(x)|$ for $x \in [1,3]$. We have $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, hence

$$|f''(x)| = \left|\frac{2}{x^3}\right| \leq \frac{2}{1^3} = 2 \text{ for } x \in [1,3].$$

This is our M!

Therefore, $|T_5| \leq \frac{8M}{300} = \frac{16}{300} \approx 0.0533.$

$\boxed{\text{Note: We showed earlier that } |T_5| \approx 0.0117\text{, which is indeed } \leq 0.0533\text{!}}$

Q: In the last example, how big could the error be if 20 subintervals are used?

A: Again, $M = \max_{1 \leq x \leq 3} |f''(x)| = 2$, so

$$|T_{20}| \leq \frac{M(b-a)^3}{12 \cdot 20^2}$$

$$= \frac{2 \cdot (3-1)^3}{12 \cdot 400} \approx \boxed{0.0033}$$

Q: How many subintervals would we need to guarantee an error of at most 0.0001?

A: With n subintervals, we have

$$|T_n| \leq \frac{2 \cdot (3-1)^3}{12 n^2} = \frac{4}{3n^2}$$

$\boxed{\text{If } \frac{4}{3n^2} \leq 0.0001\text{, then } |T_n| \text{ will certainly be } \leq 0.0001\text{!}}$

$$\frac{4}{3n^2} \leq 0.0001 \quad \Rightarrow \quad \frac{4}{3} \leq 0.0001 \cdot n^2$$

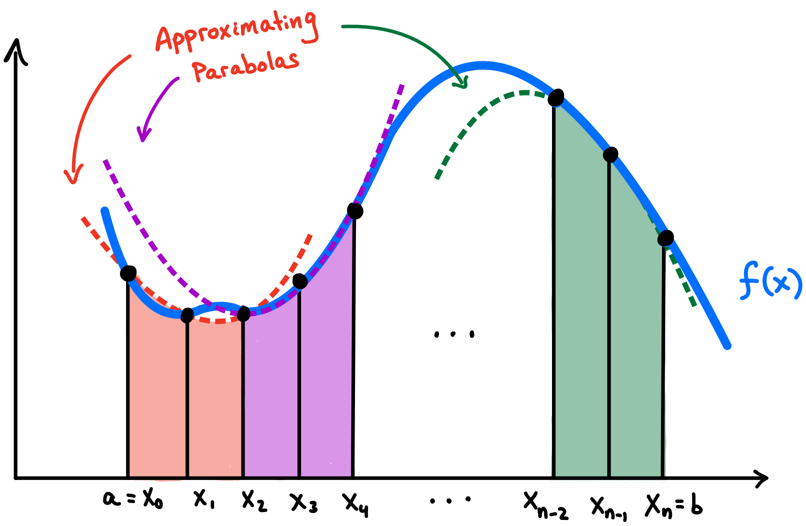
$$\Rightarrow \quad \frac{4}{3 \cdot 0.0001} \leq n^2$$

$$\Rightarrow \quad n \geq \sqrt{\frac{4}{0.0003}} \approx 115.47$$

$$\Rightarrow \quad \boxed{n \text{ should be at least } 116.}$$

3. Simpson's Rule

We'll now approximate $\int_a^b f(x)\,dx$ using parabolas!



Each parabola occupies <u>2</u> subintervals, so the number of subintervals, n, needs to be <u>even</u> in Simpson's Rule!

In this case, $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, and

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]$$

$$= \frac{\Delta x}{3}\Big[f(x_0) + f(x_n) + 4f(x_{odds}) + 2f(x_{evens})\Big]$$

Simpson's Rule!

Furthermore, the error in this approximation,

$$S_n = (\text{actual value}) - (\text{approximation})$$

satisfies the following inequality.

Error Bound for Simpson's Rule

$$\boxed{|S_n| \le \frac{M(b-a)^5}{180n^4}}$$

where M is an upper bound for $|f^{(4)}(x)|$ for $x \in [a,b]$.

Ex: (a) Use Simpson's rule with $n=6$ to approximate $\int_1^4 \ln x \, dx$

Solution: $\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = 0.5$, $x_i = a + i\Delta x = 1 + 0.5x$

$$\int_1^4 \ln x \, dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\Big]$$

$$= \frac{0.5}{3}\Big[\ln 1 + 4\ln 1.5 + 2\ln 2 + 4\ln 2.5 + 2\ln 3 + 4\ln 3.5 + \ln 4\Big]$$

$$\approx \boxed{2.545}$$

(b) Estimate the size of the error from (a).

Solution: To get M, we look at $f^{(4)}(x)$ for $f(x) = \ln x$.

$$f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = \frac{-6}{x^4}$$

Hence for $x \in [1,4]$,

$$\left|f^{(4)}(x)\right| = \left|\frac{-6}{x^4}\right| \le \frac{6}{1^4} = 6.$$

our M!

Thus,

$$|S_6| \leq \frac{M(b-a)^5}{180n^4} = \frac{6(4-1)^5}{180 \cdot 6^4} = \boxed{0.00625}$$

(c) How many subintervals, n, would we need to ensure the error from Simpson's rule is ≤ 0.0002?

<u>Solution:</u> Again, we have $M = 6$. If we were to use n subintervals, the error, S_n, would satisfy

$$|S_n| \leq \frac{M(b-a)^5}{180 \cdot n^4} = \frac{6 \cdot 3^5}{180\, n^4} = \frac{81}{10n^4}$$

We'll choose n such that $\frac{81}{10n^4} \leq 0.0002$:

$$\frac{81}{10n^4} \leq 0.0002 \quad \Rightarrow \quad \frac{81}{10 \cdot 0.0002} \leq n^4$$

$$\Rightarrow \quad n \geq \sqrt[4]{\frac{81}{0.002}} \approx 14.19$$

Since n must be even, $\boxed{n = 16}$ will do.

Additional Exercises:

Ex: Approximate $\int_0^1 (x+3)^{5/2}\,dx$ using

(a) the trapezoid rule with $n=4$

(b) Simpson's rule with $n=4$.

In each case, estimate the size of the error.

Solution: $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$ and hence

$$x_i = a + i\Delta x = \frac{i}{4} \quad \left(=0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1\right)$$

(a) $$\int_0^1 (x+3)^{5/2}\,dx \approx \frac{\Delta x}{2}\left[f(0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$$

$$= \frac{1/4}{2}\left[3^{5/2}+2\cdot 3.25^{5/2}+2\cdot 3.5^{5/2}+2\cdot 3.75^{5/2}+4^{5/2}\right]$$

$$\approx \boxed{23.25}$$

Since $f'(x) = \frac{5}{2}(x+3)^{3/2}$ and $f''(x) = \frac{15}{4}(x+3)^{1/2}$, for $x \in [0,1]$, we have

$$|f''(x)| = \left|\frac{15}{4}(x+3)^{1/2}\right| \leq \frac{15}{4}(1+3)^{1/2} = \underbrace{\frac{15}{2}}_{\text{Our } M!}$$

and hence

$$|T_4| \leq \frac{M(b-a)^3}{12n^2} = \frac{\frac{15}{2}\cdot(1-0)^3}{12\cdot 4^2} \approx \boxed{0.0391}$$

(b) Using Simpson's rule,

$$\int_0^1 (x+3)^{5/2}\,dx \approx \frac{\Delta x}{3}\left[f(0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right]$$

$$= \frac{1/4}{3}\left[3^{5/2}+4\cdot 3.25^{5/2}+2\cdot 3.5^{5/2}+4\cdot 3.75^{5/2}+4^{5/2}\right]$$

$$\approx \boxed{23.21}$$

From the derivative calculations in (a), we have

$$f'''(x) = \frac{15}{8}(x+3)^{-1/2} \quad \text{and} \quad f^{(4)}(x) = -\frac{15}{16}(x+3)^{-3/2},$$

hence, for $x \in [0,1]$,

$$\left|f^{(4)}(x)\right| = \left|\frac{-15}{16}(x+3)^{-3/2}\right| = \frac{15}{16(x+3)^{3/2}} \leq \frac{15}{16(0+3)^{3/2}} \approx \underbrace{0.5413}_{\text{Our } M!}.$$

Therefore, the error satisfies

$$|S_4| \leq \frac{M(b-a)^5}{180 \cdot n^4} = \frac{0.5413(1-0)^5}{180 \cdot 4^4} \approx \boxed{0.00001175.}$$