§ 10.5 – New Taylor Series from Old

We can use the Maclaurin series for e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$ to quickly obtain Taylor and Maclaurin series for related functions!

Ex: Find the Maclaurin series for $f(x)$ and its radius and interval of convergence.

(a) $f(x) = xe^x$

Solution: We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in (-\infty, \infty)$.

Thus,

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}},$$ and

this series will also converge for $x \in (-\infty, \infty)$.

So $\boxed{I = (-\infty, \infty), \quad R = \infty.}$

(b) $f(x) = \sin(x^2)$.

Solution: We know $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for $x \in (-\infty, \infty)$. Hence

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}}$$

and this series will also converge for $x \in (-\infty, \infty)$.

Thus, $\boxed{I = (-\infty, \infty), \quad R = \infty}$

(c) $f(x) = \dfrac{2}{3-x}$

Solution: We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

Thus, $\frac{2}{3-x} = \frac{2}{3}\left(\frac{1}{1-\frac{x}{3}}\right)$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

For convergence we need $\left|\frac{x}{3}\right| < 1$

$\Rightarrow |x| < 3$

$\Rightarrow x \in (-3, 3)$

$$= \boxed{\sum_{n=0}^{\infty} \frac{2 \cdot x^n}{3^{n+1}}}.$$

This series will converge for x in $\boxed{I = (-3,3), \quad \text{so } R = 3.}$

(d) $f(x) = \dfrac{x^3}{4+x^2}$

Solution: Again, note that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

Thus,

$$\frac{x^3}{4+x^2} = \frac{x^3}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right)$$

$$= \frac{x^3}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right)$$

For convergence, we need $\left|\frac{-x^2}{4}\right| < 1$

$\Rightarrow |x|^2 < 4$

$\Rightarrow |x| < 2$

$\Rightarrow x \in (-2,2)$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} \left(\frac{-x^2}{4}\right)^n$$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{4^{n+1}}}$$

This series converges for x in $\boxed{I = (-2,2), \quad \text{so } R = 2.}$

Ex: Find the Taylor series for $f(x) = e^x$ centred at $x = 4$.

Solution 1: Note that $f^{(n)}(x) = e^x$ for all n, and hence $f^{(n)}(4) = e^4$ for all n. Thus, from the definition of the Taylor series, we obtain

$$\boxed{\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n &= \sum_{n=0}^{\infty} \frac{e^4}{n!}(x-4)^n \\ &= e^4 + e^4(x-4) + \frac{e^4}{2!}(x-4)^2 + \cdots\end{aligned}}$$

Solution 2: Starting with the Maclaurin series for $f(x) = e^x$, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{x-4} = \sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$$

$$\Rightarrow \frac{e^x}{e^4} = \sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$$

$$\xRightarrow{\cdot e^4} \boxed{e^x = \sum_{n=0}^{\infty} \frac{e^4}{n!}(x-4)^n}$$

We can also obtain new Taylor series by differentiating or integrating a Taylor series term-by-term. These operations won't change the radius of convergence!

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with radius of convergence $R > 0$, then

(i) $f'(x) = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$

Note that differentiation kills the $n=0$ term (i.e., the constant term!)

(ii) $\int f(x)\, dx = \left(\sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} \right) + C$

and both of these series have radius of convergence R.

Note: While differentiating or integrating a power series won't change its radius, they may change the interval!

We need to re-check convergence at the endpoints!

Ex: Find the Maclaurin series and interval of convergence.

(a) $f(x) = \frac{1}{(1-x)^2}$.

Solution: Note that

$$\frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]' = \left[\sum_{n=0}^{\infty} x^n\right]' = \boxed{\sum_{n=1}^{\infty} n x^{n-1}}$$

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$,

So too does our new series. Thus, we at least have convergence for $x \in (-1,1)$. We check the endpoints.

$$x = 1 \Rightarrow \sum_{n=1}^{\infty} n(1)^{n-1} = \sum_{n=1}^{\infty} n$$

Diverges by the divergence test!

$$x = -1 \Rightarrow \sum_{n=1}^{\infty} n(-1)^{n-1}$$

Diverges by the divergence test!

$\therefore$ The interval of convergence is $\boxed{I = (-1,1)}$.

(b) $f(x) = \frac{1}{(1-x)^3}$

Solution: We have

Radius $R = 1$

$$\frac{2}{(1-x)^3} = \left[\frac{1}{(1-x)^2}\right]' = \left[\sum_{n=1}^{\infty} n x^{n-1}\right]' = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

Hence $\boxed{\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}}$ with radius $R = 1$.

Exercise: Check convergence at $x = \pm 1$. You should find that $\boxed{I = (-1, 1).}$

(c) $f(x) = \arctan(x)$

Solution: We'll start by finding the Maclaurin series for $\frac{1}{1+x^2}$. We have

For convergence, we need

$|-x^2| < 1 \Rightarrow |x|^2 < 1$

$\Rightarrow |x| < 1 \quad (R=1)$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Thus, $\arctan(x) = \displaystyle\int \frac{1}{1+x^2}\,dx$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \right) + C$$

We can find C by plugging in $x=0$ (or in general, $x=$ centre):

$$\arctan(0) = \left(\sum_{n=0}^{\infty} \overbrace{\cancel{\frac{(-1)^n\, 0^{2n+1}}{2n+1}}}^{=0} \right) + C \quad \Rightarrow \quad \underline{C = \arctan(0) = 0.}$$

Therefore,

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

The radius is still $R=1$, so it converges for $x \in (-1,1)$.

Let's check the endpoints!

$$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n \, 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$ (converges by the AST!)

$$x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$ (converges by the AST!)

-1 to odd power is just -1

Thus, the interval of convergence is $\boxed{I = [-1, 1].}$

Additional Exercises

1. Use integration to show that

$$\boxed{\ln|1+x| = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$$

with interval of convergence $I = (-1, 1]$.

2. Find the Maclaurin series for $f(x) = \dfrac{x^2}{(3+x)^2}$ and its interval of convergence.

Solutions:

(a) First, note that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

with radius of convergence $R=1$. Hence,

$$\begin{aligned} \ln|1+x| &= \int \frac{1}{1+x}\, dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right) + C \end{aligned}$$

also with $R=1$. We find C by setting $x=0$:

$$\ln|1+0| = \sum_{n=0}^{\infty} \cancelto{=0}{\frac{(-1)^n 0^{n+1}}{n+1}} + C \Rightarrow C = \ln 1 = 0.$$

We therefore have

$$\ln|1+x| = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Lastly, we check convergence at $x = \pm 1$.

$\underline{x = 1}$ $\Rightarrow$ $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$ (converges by AST)

$\underline{x = -1}$ $\Rightarrow$ $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1}$$ Always $= -1$

$$= \underbrace{- \sum_{n=0}^{\infty} \frac{1}{n+1}}_{\text{harmonic series}} \Rightarrow \text{divergent}$$

Thus, $I = (-1, 1]$, as required.

(b) Note that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with radius $R = 1$.

$$\Rightarrow \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{with } R = 1$$

$$\Rightarrow \frac{x^2}{(3+x)^2} = \frac{x^2}{3^2 \left(1 + \frac{x}{3}\right)^2} = \frac{x^2}{9} \cdot \frac{1}{\left(1 - \left(\frac{-x}{3}\right)\right)^2}$$

Replace x with $\frac{-x}{3}$

$$= \frac{x^2}{9} \sum_{n=1}^{\infty} n \left(\frac{-x}{3}\right)^{n-1}$$

For convergence, need $\left|\frac{-x}{3}\right| < 1$

$\Rightarrow |x| < 3$

$\Rightarrow x \in (-3, 3)$.

$$= \frac{x^2}{9} \sum_{n=1}^{\infty} \frac{n(-1)^{n-1} x^{n-1}}{3^{n-1}}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{n(-1)^{n-1} x^{n+1}}{3^{n+1}}}$$

<u>Exercise:</u> Check convergence at the endpoints of $(-3,3)$. You should find that the series diverges in each case, hence

$$\boxed{I = (-3,3).}$$