§ 15.3 – First Order Linear Differential Equations

A first order linear DE has the form

$$\boxed{A(x)\cdot y' + B(x)\cdot y = C(x), \quad \text{where } A(x) \not\equiv 0}$$

e.g. $xy' + 2y = x$ ⟵ linear!

$y' + xy^2 = 1$, $y \cdot y' = x$ ⟵ non-linear!

Alternatively, dividing by $A(x)$, it can be written as

$$\boxed{y' + P(x)\cdot y = Q(x)}$$

Let's start with an example, then we'll explore the general strategy!

Ex: Let's solve the linear DE $y' + \frac{2}{x}\cdot y = 1$

For this DE, multiply both sides by x^2!

We get $\underbrace{x^2 y' + 2x \cdot y}_{} = x^2. \implies (x^2 y)' = x^2$

Note: LHS is now $(x^2y)'$ by the product rule!

Now integrate both sides and solve for y:

$$\int (x^2y)'\,dx = \int x^2\,dx \implies x^2 y = \frac{x^3}{3} + C$$

$$\implies \boxed{y = \frac{x}{3} + \frac{C}{x^2}, \quad C \in \mathbb{R}}$$

In general, we solve $y' + P(x)y = Q(x)$ by multiplying both sides by

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

We call this an integrating factor.

In our example: $y' + \frac{2}{x} \cdot y = 1$

P(x)

$$\implies \mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2 \,!$$

Note: You can write $\ln(x)$ instead of $\ln|x|$ when calculating $\mu(x)$. You can also omit the "$+C$". The result will be the same when μ is multiplied into the DE!

Why is it helpful to multiply by $\mu(x)$??

Well... since

$$\mu'(x) = \underbrace{e^{\int P(x)\,dx}}_{=\mu(x)} \cdot \underbrace{\left(\int P(x)\,dx\right)'}_{=P(x)} = \mu(x)\,P(x),$$

the DE $y' + P(x)\,y = Q(x)$ becomes

$$\mu(x)\cdot y' + \underbrace{\mu(x)\,P(x)}_{=\mu'(x)}\cdot y = \mu(x)\,Q(x)$$

$$\Rightarrow \quad \mu(x)\cdot y' + \mu'(x)\cdot y = \mu(x)\,Q(x)$$

$$\Rightarrow \quad \left[\mu(x)\cdot y\right]' = \mu(x)\,Q(x)$$

Now integrate both sides and solve for y!

To solve $A(x)y' + B(x)y = C(x)$:

1. Write the DE as $y' + P(x)y = Q(x)$
2. Multiply both sides by $\mu(x) = e^{\int P(x)\,dx}$.
3. Rewrite the LHS as $\left[\mu(x)\cdot y\right]'$
4. Integrate both sides with respect to x.
5. Solve for y.

Ex: Solve $xy' - y = x^2\cos x$

Solution: First divide by x to get

$$y' - \frac{1}{x}\cdot y = x\cdot\cos x$$

This is linear with $P(x) = \frac{-1}{x}$. We multiply by

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}.$$

to get

$$y' - \frac{1}{x}\cdot y = x\cdot\cos x \implies \frac{1}{x}y' - \frac{1}{x^2}y = \cos x$$

$$\implies \left[\frac{1}{x}\cdot y\right]' = \cos x$$

Integrate!
$$\implies \frac{1}{x}\cdot y = \sin x + C$$

$$\implies \boxed{y = x\cdot\sin x + Cx, \quad C\in\mathbb{R}}$$

Ex: Solve the IVP $y' + 2xy = 4x$, $y(0) = 7$.

Solution: This is a linear DE with $P(x) = 2x$, hence we multiply by $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$. We have

$$y' + 2xy = x \implies e^{x^2}y' + 2xe^{x^2}y = 4xe^{x^2}$$

$$\implies \left[e^{x^2}y\right]' = 4xe^{x^2}$$

Integrate!
$$\implies e^{x^2}y = \int 4xe^{x^2}\,dx \quad \left(\begin{array}{l}\text{let } u = x^2\\ du = 2x\,dx\end{array}\right)$$

$$\implies e^{x^2}y = 2\int e^u\,du = 2e^{x^2} + C$$

$$\Rightarrow \quad y = 2 + \frac{c}{e^{x^2}}, \quad c \in \mathbb{R}$$

Using the initial condition $y(0) = 7$, we get

$$7 = 2 + \frac{c}{e^{(0)^2}} = 2 + \frac{c}{1} \quad \Rightarrow \quad \underline{c = 5}$$

Thus, $\boxed{y = 2 + \frac{5}{e^{x^2}}}$

Note: The above DE is also separable! Try solving the problem using the methods of §15.2!

Additional Exercise:

Solve the IVP $2xy' + y = 2x^2$, $y(1) = 0$.

Solution: Divide by $2x$ to get

$$y' + \frac{1}{2x} y = x,$$

which is linear with $P(x) = \frac{1}{2x}$. We multiply by

$$\mu(x) = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\ln(x)} = e^{\ln(x^{1/2})} = \underline{\sqrt{x}}$$

to get

$$\sqrt{x}\,y' + \frac{1}{2\sqrt{x}}\cdot y = x\cdot\sqrt{x} \implies \left[\sqrt{x}\cdot y\right]' = x^{3/2}$$

$$\implies \sqrt{x}\cdot y = \int x^{3/2}\,dx$$

$$\implies \sqrt{x}\cdot y = \frac{2}{5}x^{5/2} + C$$

Using $y(1) = 0$, we get

$$\underbrace{\sqrt{1}\cdot 0}_{=0} = \underbrace{\frac{2}{5}(1)^{5/2}}_{=2/5} + C \implies \underline{C = -2/5}$$

Therefore,

$$\sqrt{x}\cdot y = \frac{2}{5}x^{5/2} - \frac{2}{5} \implies \boxed{y = \frac{2}{5}x^{2} - \frac{2}{5\sqrt{x}}}$$