

§10.9 - Infinite Series

Let $\{a_n\}$ be a sequence. An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \dots = \sum_{n=1}^{\infty} a_n$$

Ex: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

Sum = 0.75

Sum = 0.875

Sum = 0.9375

⋮

Perhaps these terms
add to 1? Hard
to tell...

Let's formalize the ideas from the examples above.

Definition: Given a series $\sum_{n=1}^{\infty} a_n$, we define its

N^{th} partial sum by

$$S_N = a_1 + a_2 + \dots + a_N = \sum_{n=1}^N a_n$$

If $\lim_{N \rightarrow \infty} S_N = S$ for some (finite) real number S , then

we say $\sum_{n=1}^{\infty} a_n$ converges to S and write

$$\sum_{n=1}^{\infty} a_n = S.$$

If instead $\lim_{N \rightarrow \infty} S_N$ DNE (i.e., doesn't approach anything

or is $\pm\infty$), we say $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: For $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$, we have

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \quad \dots$$

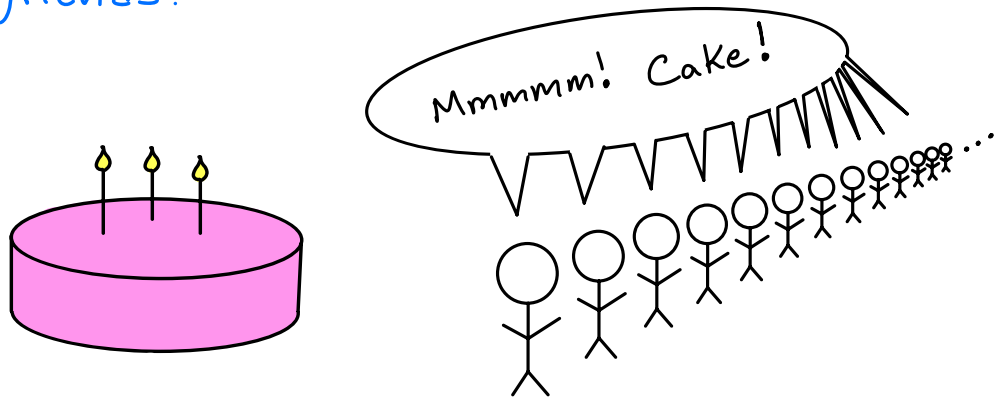
Noticing a pattern, we have $S_N = \frac{2^N - 1}{2^N}$, hence

$$\lim_{n \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \frac{2^N - 1}{2^N} = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{2^N} \right) = 1.$$

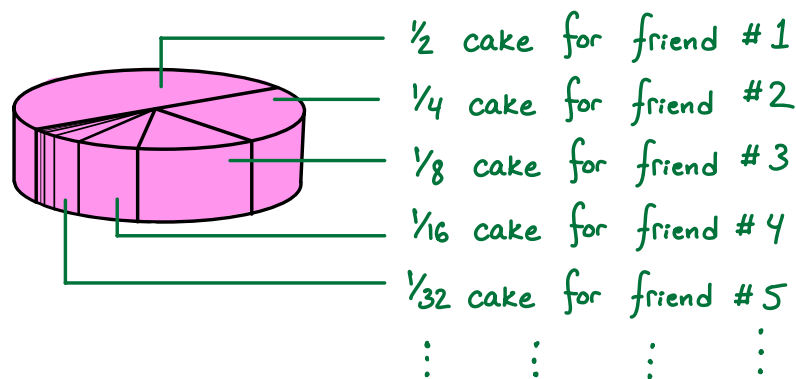
$$\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \quad (\text{convergent!})$$

You can think about this particular sum as follows:

Imagine sharing a cake with your infinitely many friends.



We'll divide up the cake in the following way:



In total, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$ cake!

Ex: For $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$, we have

$$S_0 = 1,$$

$$S_1 = 1 - 1 = 0,$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0, \dots$$

In general, $S_N = \begin{cases} 1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd.} \end{cases}$

Thus, $\lim_{N \rightarrow \infty} S_N$ DNE, so

$\sum_{n=0}^{\infty} (-1)^n$ diverges.

Ex: For $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$, we have

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3} \right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4} \right) = 1 - \frac{1}{4}$$

\vdots

\vdots

In general,

$$S_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots$$

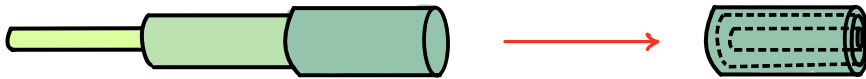
$$\dots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right) = \boxed{1 - \frac{1}{N+1}}$$

Thus, $\lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1}\right) = 1$, hence

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 \quad (\text{convergent!})$$

Note: $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ referred to as a telescoping series,

since the middle terms "collapse" when we compute S_N !



Ex: It turns out that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, even though the terms $\frac{1}{n}$ are very tiny

when n is large. Let's see why!

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} \geq 1 + 2\left(\frac{1}{2}\right)$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} \geq 1 + 3\left(\frac{1}{2}\right)$$

$$S_{16} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}} \\ \geq 1 + 4\left(\frac{1}{2}\right)$$

...

In general,

$$S_{2^N} \geq \underbrace{1 + N\left(\frac{1}{2}\right)}_{\rightarrow \infty \text{ as } N \rightarrow \infty} \text{ for all } N.$$

Thus, $\lim_{N \rightarrow \infty} S_N = \infty$, meaning $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. ■