§ 15.1 – Introduction to Differential Equations

A differential equation (DE) is an equation involving an unknown function and its derivatives. The equation may contain $x, y, y', y'', \dots, y^{(n)}$.

Typically, our goal is to solve the equation for the function(s) y.

Definition: The order of a DE is the order of the highest derivative that appears.

Ex: $\underbrace{y' = x}$ is a DE of order 1.

"which function(s) y differentiates to x?"

This DE isn't very interesting ... we can solve it using integration:

$$y' = x \implies y = \int x\,dx = \boxed{\frac{x^2}{2} + C}$$

Ex: $\frac{dy}{dx} = y$ is a more interesting DE of order 1.

"Which function(s) y differentiates to itself?"

Solutions to this DE include $y = e^x$, but also $y = 0$, $y = 2e^x$, $y = \pi e^x$, ... In general, $\boxed{y = Ce^x,\ C \in \mathbb{R}}$

The complete set of solutions to a DE (including any arbitrary constants) is called the general solution.

In our examples, we refer to the general solutions

$$y = \frac{x^2}{2} + C,\ C \in \mathbb{R} \quad \text{and} \quad y = Ce^x,\ C \in \mathbb{R}$$

as one-parameter families of solutions (since each involves exactly one arbitrary constant.)

More examples of first order DEs:

$$\frac{dy}{dx} = y^2, \qquad \frac{dy}{dx} = xy, \qquad y' + 2y = e^x$$

(Already much harder to "see" solutions...)

We'll learn how to solve all of the above examples in MATH 118. First order DEs will be our focus, but you'll learn how to solve more complicated DEs in later courses:

AE	CHE	CIVE	ENVE	GEOE	MGTE	ME	MTE
AE 223	MATH 218	CIVE 222	ENVE 223	GEOE 223	MSCI 271	ME 203 ME 303	MTE 202 MTE 204

Although finding solutions to DEs can be hard, it's usually easy to verify whether a specific function is a solution or not.

Ex: Is $y = \sqrt{x^2+1}$ a solution to the DE

$$\frac{dy}{dx} = \frac{x}{y} \, ?$$

Solution: Let's compute the left and right hand sides.

LHS: $\frac{dy}{dx} = \frac{d}{dx}\sqrt{x^2+1} = \cancel{2}x \cdot \frac{1}{\cancel{2}}(x^2+1)^{-1/2} = \frac{x}{\sqrt{x^2+1}}$

RHS: $\frac{x}{y} = \frac{x}{\sqrt{x^2+1}}$

Equal!

$\boxed{\text{Yes, } y = \sqrt{x^2+1} \text{ is a solution.}}$

Now let's see how we could have arrived at this solution ourselves!