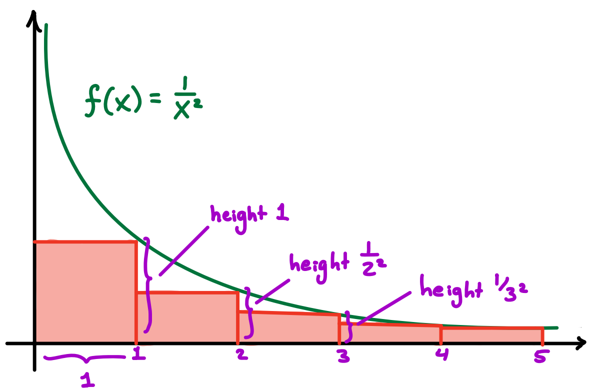
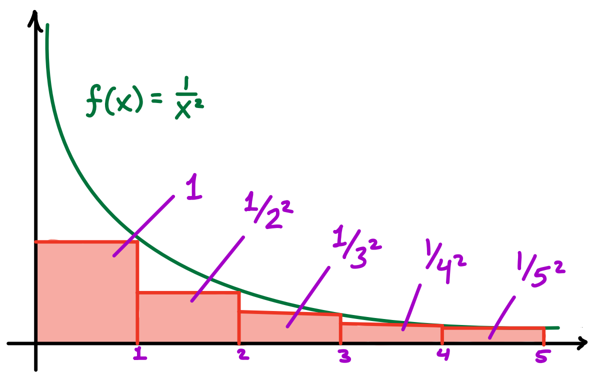
② The Integral Test

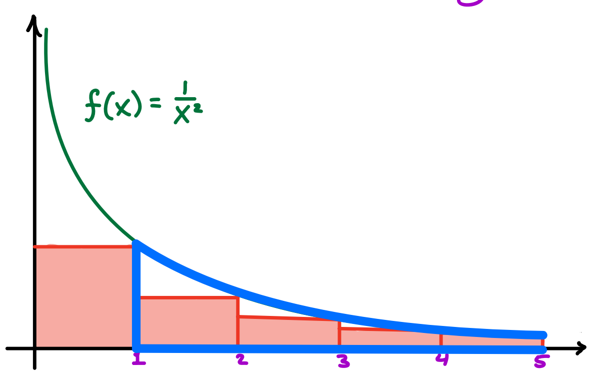
We'll introduce this test by studying the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Idea: Think of each term $\frac{1}{n^2}$ like the area of a $\frac{1}{n^2} \times 1$ rectangle.





Thus, the shaded area is exactly $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Is this area finite? Well... if the area under $y = \frac{1}{x^2}$ from 1 to ∞ is finite, then the shaded area must also be finite!



Area under $y = \frac{1}{x^2}$ $= \int_1^\infty \frac{1}{x^2}\, dx$

$$= \lim_{t \to \infty} \int_1^t \frac{1}{x^2}\, dx$$

$$= \lim_{t \to \infty} \left[\cancelto{0}{\frac{-1}{t}} - \frac{-1}{1} \right] = 1 \quad (\text{finite!})$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must also be finite (i.e., convergent!)

The above argument is the main idea behind...

The Integral Test

Suppose $f(x)$ is continuous, positive, and decreasing on an interval $[k, \infty)$.

(i) If $\int_k^\infty f(x)\, dx$ converges, then $\sum_{n=k}^{\infty} f(n)$ converges.

(ii) If $\int_k^\infty f(x)\, dx$ diverges, then $\sum_{n=k}^{\infty} f(n)$ diverges.

Ex: Does $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln(n)}$ converge or diverge?

Solution: Let $f(x) = \frac{1}{x \cdot \ln(x)}$. On $[2, \infty)$,

- f is continuous (since f is a quotient of non-zero continuous functions)
- f is positive (since $x > 0$ and $\ln x > 0$ on $[2,\infty)$)
- f is decreasing (since x and $\ln x$ are increasing)

Hence, the assumptions of the integral test are satisfied on $[2, \infty)$. We have

$$\int_2^{\infty} \frac{1}{x \cdot \ln x}\, dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \cdot \ln x}\, dx$$

$$= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\, du \qquad \left(\begin{array}{l} u = \ln x \\ du = 1/x \, dx \end{array} \right)$$

$$= \lim_{t \to \infty} \ln|\ln t| - \ln|\ln 2| = \infty$$

Since $\int_2^{\infty} \frac{1}{x \cdot \ln x}\, dx$ diverges, $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ diverges too.

Ex: Does $\sum_{n=0}^{\infty} \frac{n}{n^2+6}$ converge or diverge?

Solution: Let's try the divergence test!

$$\lim_{n\to\infty} \frac{n}{n^2+6} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{2n} = 0 \quad \Rightarrow \quad \text{No conclusion!}$$

Hmm... okay, let's try the integral test!

Let $f(x) = \frac{x}{x^2+6}$. Note that f is continuous (since f is a rational function and $x^2+6 \neq 0$) and positive for $x > 0$.

Is f decreasing? Not clear — let's compute $f'(x)$!

$$f(x) = \frac{x}{x^2+6} \quad \Rightarrow \quad f'(x) = \frac{(x^2+6)\cdot 1 - x\cdot(2x)}{(x^2+6)^2} = \frac{6-x^2}{(x^2+6)^2}$$

quotient rule!

$$\Rightarrow \quad f'(x) < 0 \quad \text{when} \quad 6 - x^2 < 0$$

$$\Rightarrow \quad f'(x) < 0 \quad \text{when} \quad x > \sqrt{6}$$

$$\Rightarrow \quad f \text{ is decreasing for } x > \sqrt{6} \approx 2.45$$

$\therefore$ The assumptions of the integral test are satisfied on $[3, \infty)$.

We have $\int_3^\infty \frac{x}{x^2+6}\,dx = \infty$ (exercise!) and therefore

$\sum_{n=3}^{\infty} \frac{n}{n^2+6}$ diverges by the integral test. Consequently,

$$\sum_{n=0}^{\infty} \frac{n}{n^2+6} = \overbrace{\sum_{n=0}^{2} \frac{n}{n^2+6}}^{\text{finite}} + \overbrace{\sum_{n=3}^{\infty} \frac{n}{n^2+6}}^{\text{divergent}}$$

is divergent as well.

Remark: When checking the convergence/divergence of series, we can look past (i.e., ignore) any finite number of terms. Convergence/divergence is determined by the infinitely many terms at the end of the series!

The integral test can be used to test the convergence of series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p = \text{constant}),$$

which we refer to as a p-series.

We will summarize this as its own test!

The p-Series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Ex: $$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

This is a convergent p-series, since $p = 3 > 1$.

Ex: $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

This is a divergent p-series, since $p = \frac{1}{2} \leq 1$.