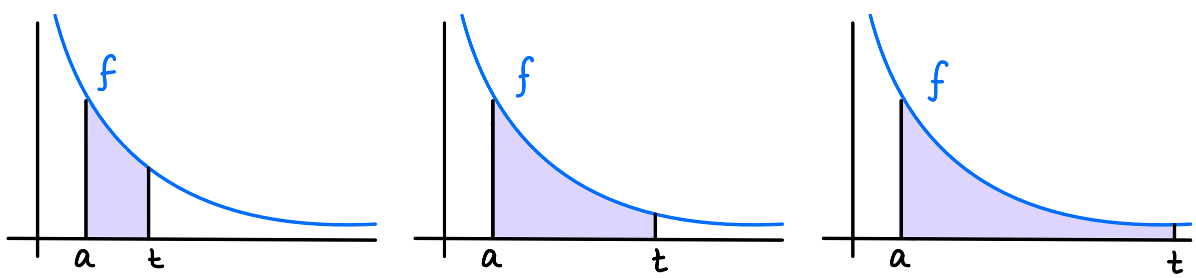
§7.10 - Improper Integrals

So far we have only examined integrals of continuous functions over finite intervals. In this section we'll learn how to handle integrals of functions with an infinite discontinuity (i.e., a vertical asymptote) and integrals over infinite domains. Integrals of these types are known as improper integrals.

① Infinite Domains

Definition: We define

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$



and similarly,

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx$$

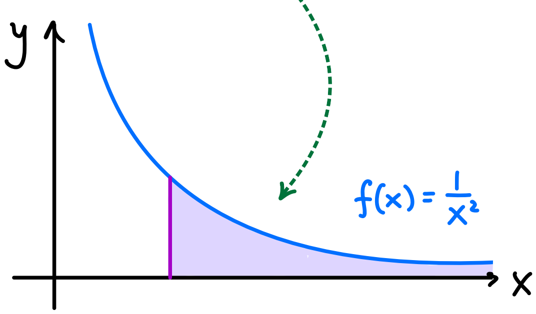
The integral converges if the limit exists. It diverges if the limit DNE (i.e., doesn't approach anything or is $\pm\infty$).

Examples:

(a) $$\int_{1}^{\infty} \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^2}\,dx = \lim_{t \to \infty} \left[\frac{-1}{x} \right]_{1}^{t}$$

$$= \lim_{t \to \infty} \left[\cancelto{0}{\frac{-1}{t}} - \left(\frac{-1}{1} \right) \right]$$

$$= 1$$



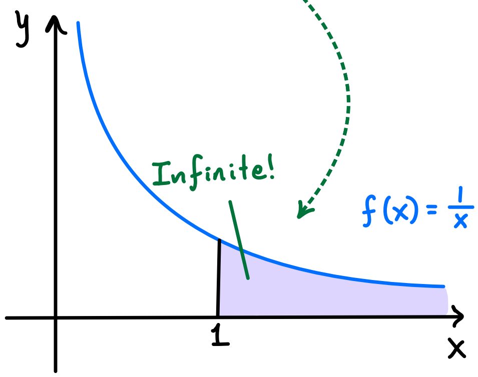
$\therefore$ Integral converges!

So the area is finite (and equal to 1!)

(b) $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\Big[\ln|x|\Big]_1^t$

$$= \lim_{t\to\infty} \underbrace{\ln|t|}_{\to\infty} - \ln 1$$

$$= \infty$$



$\therefore$ Integral <u>diverges!</u>

For which values of p does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Well... From (a), $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ diverges when $p=1$.

For $p \neq 1$, we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

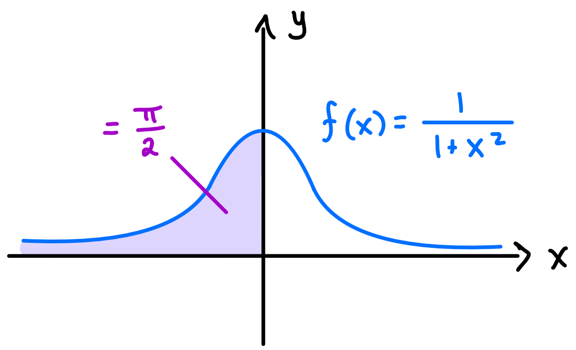
$$= \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)$$

Note that $t^{-p+1} \to \infty$ if $-p+1 > 0$ (i.e., $p<1$), in which case the integral diverges. But if $-p+1<0$ (i.e., $p>1$), then $t^{-p+1} \to 0$ and the integral converges. In summary...

Theorem [Convergence of p-Integrals]

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p \le 1$.

(c) $$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \Big[\arctan x\Big]_t^0$$



$$= \lim_{t\to-\infty}\Big[\underbrace{\arctan(0)}_{=0} - \underbrace{\arctan(t)}_{\to -\pi/2}\Big]$$

$$= \boxed{\pi/2}$$

Thus, the integral converges!

We also define

(Deal with each infinity separately!)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \lim_{t\to\infty} \int_{t}^{0} f(x)\,dx + \lim_{s\to\infty} \int_{0}^{s} f(x)\,dx$$

If both limits exist, we say $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

If even one limit DNE, $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

Ex: $$\int_{-\infty}^{\infty} x\cos(x^2)\,dx = \lim_{t\to-\infty} \int_{t}^{0} x\cos(x^2)\,dx + \underbrace{\lim_{s\to\infty} \int_{0}^{s} x\cos(x^2)\,dx}_{\text{Let's try computing this first.}}$$

$$\lim_{s\to\infty} \int_{0}^{s} x\cos(x^2)\,dx = \lim_{s\to\infty} \frac{1}{2}\int_{1}^{s^2} \cos(u)\,du$$

$u = x^2$
$du = 2x\,dx$

$$= \lim_{s\to\infty} \frac{1}{2}\left[\sin(u)\right]_{0}^{s^2}$$

$$= \lim_{s \to \infty} \frac{1}{2} \sin(s^2)$$

Oscillates, doesn't approach anything

$$\Rightarrow \lim_{s \to \infty} \int_0^s x\cos(x^2)\,dx \quad \text{DNE}$$

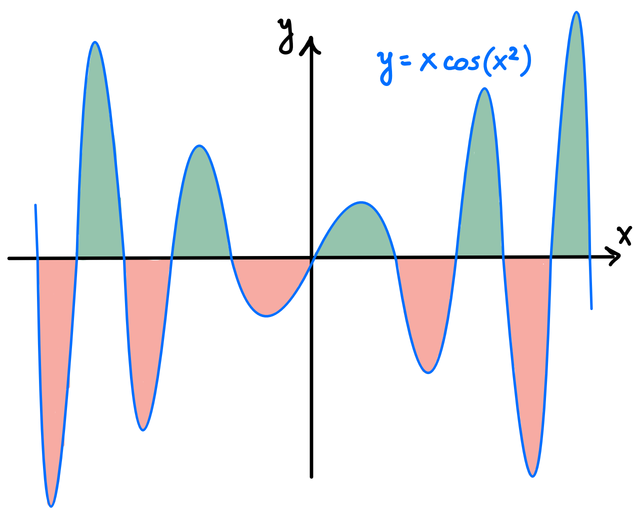
$$\Rightarrow \int_{-\infty}^{\infty} x\cos(x^2)\,dx \quad \underline{\text{diverges.}}$$

Recall: If even one of the limits DNE, the integral diverges.

In this case, there is no need to check the other limit!

CAUTION!

For improper integrals, it's NOT true that the positive and negative areas "cancel" to give $\int_{-\infty}^{\infty} x\cos(x^2)\,dx = 0$.

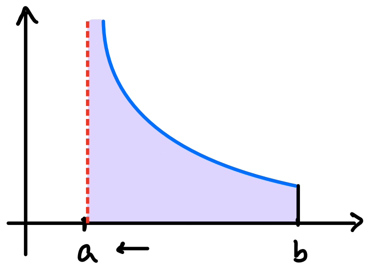


Our definition requires both $\int_0^{\infty} x\cos(x^2)\,dx$ and $\int_{-\infty}^{0} x\cos(x^2)\,dx$ to converge!

(II) Integrands with an Infinite Discontinuity

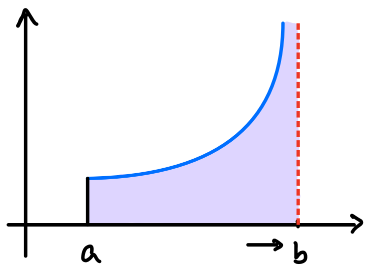
(i) If f has an infinite discontinuity at $x = a$, we use

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx}$$



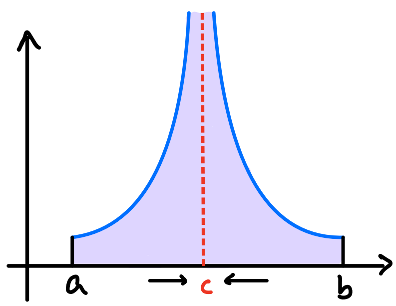
(ii) If f has an infinite discontinuity at $x = b$, we use

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx}$$



(iii) If f is discontinuous at $x = c$ with $a < c < b$, we use

$$\boxed{\begin{aligned} \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{s \to c^+} \int_s^b f(x)\,dx \end{aligned}}$$



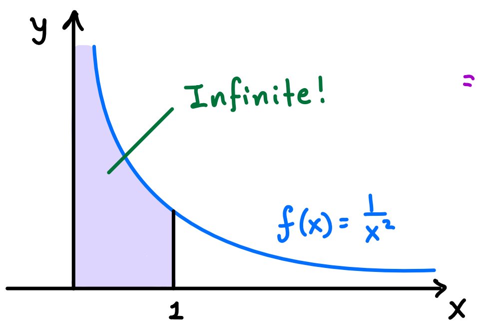
The integral converges if (all) its limit(s) exist.

If even one limit DNE, the integral diverges.

Examples:

problem at $x=0$

(a) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx \;=\; \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx$



$$= \lim_{t\to 0^+}\left[\frac{1}{t}-\frac{1}{1}\right] = \infty$$

Thus, the integral diverges.

problem at 2, but we'll start with a substitution.

(b) $\displaystyle\int_1^2 \frac{x}{x^2-4}\,dx$

Let $u=x^2-4$, $x=2 \Rightarrow u=0$

$du=2x\,dx$, $x=1 \Rightarrow u=-3$

problem at 0

$$= \int_{-3}^{0} \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$$

Limit DNE!

$$= \lim_{t\to 0^-}\frac{1}{2}\int_{-3}^{t}\frac{1}{u}\,du = \lim_{t\to 0^-}\frac{1}{2}\left[\cancelto{-\infty}{\ln|t|} - \ln|-3|\right] = -\infty$$

$\Rightarrow$ Integral diverges!

problem at $x = 1$.

(c) $$\int_0^4 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t \to 1^-} \int_0^t (x-1)^{-1/3}\,dx + \lim_{s \to 1^+} \int_s^4 (x-1)^{-1/3}\,dx$$

Let $u = x-1$

$du = dx$

$$= \lim_{t \to 1^-} \int_{-1}^{t-1} u^{-1/3}\,du + \lim_{s \to 1^+} \int_{s-1}^{3} u^{-1/3}\,du$$

$$= \lim_{t \to 1^-} \frac{3}{2}\left[\underbrace{(t-1)^{2/3}}_{\to 0} - \underbrace{(-1)^{2/3}}_{=1} \right] + \lim_{s \to 1^+} \frac{3}{2}\left[3^{2/3} - \underbrace{(s-1)^{2/3}}_{\to 0} \right]$$

$$= \frac{3}{2}\left[3^{2/3} - 1 \right]$$ Finite!

$\therefore$ Integral <u>converges.</u>