Geometric Series

A geometric series is a series of the form

$$\sum_{n=0}^{\infty} a \cdot r^n = a + ar + ar^2 + ar^3 + \cdots$$

$$\cdot r \qquad \cdot r \qquad \cdot r$$

where a, r are constants. We refer to r as the common ratio.

Ex: $\displaystyle\sum_{n=0}^{\infty} 2 \cdot \left(\frac{3}{2}\right)^n = 2 + 2\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}\right)^2 + \cdots$

This is a geometric series with $a = 2$, $r = 3/2$.

Ex: $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{-1}{5}\right)^n = 3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \cdots$

This is a geometric series with $a = 3$, $r = \frac{-1}{5}$.

Question: When does a geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ converge? When does it diverge?

Answer: Let's consider a few cases...

Case I: $r = 1$

We have $\displaystyle\sum_{n=0}^{\infty} ar^n = \underbrace{a + a + a + \cdots}_{\text{sum blows up!}} \Rightarrow$ divergent!

Case II: $r = -1$

We have $\displaystyle\sum_{n=0}^{\infty} ar^n = \underbrace{a - a + a - a + \cdots}_{\text{Sum never stabilizes!}} \Rightarrow$ divergent

Case III: $r \neq \pm 1$

Let's examine the partial sum S_N. We have

$$\text{①} \qquad S_N = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^N}$$

$$\text{②} \qquad r \cdot S_N = \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^N} + ar^{N+1}$$

$$\text{①} - \text{②} \qquad S_N - r \cdot S_N = a - ar^{N+1}$$

$$\Rightarrow (1-r)\, S_N = a\left(1 - r^{N+1}\right)$$

$$\Rightarrow \boxed{S_N = \frac{a\left(1-r^{N+1}\right)}{1-r}}$$

To check convergence, we consider $\displaystyle\lim_{N\to\infty} S_N$:

If $r > 1$ or $r < -1$, then r^{N+1} blows up as $N \to \infty$;

but if $-1 < r < 1$, then $r^{N+1} \longrightarrow 0$ as $N \longrightarrow \infty$. Hence,

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-\overbrace{r^{N+1}}^{\to 0 \text{ if } |r|<1})}{1-r} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{DNE} & \text{if } |r| > 1. \end{cases}$$

The Geometric Series Test

Consider the geometric series $\sum_{n=0}^{\infty} ar^n$.

(i) If $|r| < 1$, then the series converges. In particular, $\boxed{\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}}$

(ii) If $|r| \geq 1$, then the series diverges.

Ex: $\sum_{n=0}^{\infty} 2\cdot\left(\frac{3}{2}\right)^n$ is geometric with $a=2$, $r = 3/2$.

Since $|r| \geq 1$, this series diverges!

Ex: $\sum_{n=0}^{\infty} 3\left(\frac{-1}{5}\right)^n$ is geometric with $a=3$, $r = -1/5$.

Since $|r| < 1$, this series converges. Specifically,

$$\sum_{n=0}^{\infty} 3\left(\frac{-1}{5}\right)^n = \frac{a}{1-r} = \frac{3}{1-(-1/5)} = \frac{3}{6/5} = \boxed{\frac{5}{2}}$$

Example: Let's use geometric series to show that

$$0.99999\ldots = 1 \;!$$

Solution:
$$\begin{aligned} 0.99999\ldots &= 0.9 + 0.09 + 0.009 + \cdots \\ &= 9\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right)^2 + 9\left(\frac{1}{10}\right)^3 + \cdots \\ &= \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n \end{aligned}$$

This is a geometric series with $r = \frac{1}{10}$. Since $|r| < 1$, the series converges. However, the sum is NOT

$$\frac{a}{1-r} = \frac{9}{1-\frac{1}{10}} = \frac{9}{9/10} = 10$$

The formula $\sum_{n=n_0}^{\infty} a \cdot r^n = \frac{a}{1-r}$ only works if $n_0 = 0$!

We have a couple options to get around this...

Option 1: Add and subtract terms to create a sum starting at $n_0 = 0$.

$$\sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = \underbrace{\sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n + 9\left(\frac{1}{10}\right)^0}_{} - 9\left(\frac{1}{10}\right)^0$$

Geometric, $a = 9$, $r = \frac{1}{10}$

$$= \sum_{n=0}^{\infty} 9\left(\frac{1}{10}\right)^n - 9\left(\frac{1}{10}\right)^0$$

$$= \frac{9}{1 - \frac{1}{10}} - 9$$

$$= 10 - 9$$

$$= \boxed{1}$$

Option 2: Reindex the sum to start at $n = 0$.

$$\sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = \sum_{n=0}^{\infty} 9\left(\frac{1}{10}\right)^{n+1}$$

Increased by 1 to make up for it!

Reduced by 1

$$= \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^n \longleftarrow \boxed{\text{Geometric, } a = \frac{9}{10},\ r = \frac{1}{10}}$$

$$= \frac{9/10}{1 - \frac{1}{10}}$$

$$= \frac{9/10}{9/10}$$

$$= \boxed{1}$$

<u>Ex:</u> $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{2n}}$ doesn't yet look like a geometric series ... but let's rewrite it as

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=2}^{\infty} \frac{3 \cdot 3^n}{(2^2)^n} = \underbrace{\sum_{n=2}^{\infty} 3\left(\frac{3}{4}\right)^n}_{\text{Geometric with } r = 3/4}$$

Since $|r| < 1$, the series converges. In particular,

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=2}^{\infty} 3\left(\frac{3}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty} 3 \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{27}{16} \cdot \left(\frac{3}{4}\right)^n$$

$$= \frac{27/16}{1 - 3/4} = \boxed{\frac{27}{4}}$$

§10.10 – 10.12 : Series Convergence Tests

For series that are neither geometric nor telescoping, it can be VERY hard to find a nice expression for the partial sums, S_N. As a result, it is often VERY hard to find the exact sum of such a series!

e.g. We will soon be able to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converges. But what's the sum?

$$S_2 = 1.25, \quad S_3 \approx 1.361, \quad S_4 \approx 1.424$$

Perhaps the sum is 1.5? 2? Nope! In 1735, after many prominent mathematicians failed to find the sum, Euler proved that

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

Proof: Beyond the scope of MATH 118!

From this point onward, we won't be interested in finding exact sums, but deciding whether a series converges or diverges. We have many tests for this!

① The Divergence Test

Our first test is based on the following observation:

If $\sum_{n=1}^{\infty} a_n$ has any hope of converging, the terms a_n must become small (i.e., $a_n \longrightarrow 0$).

Thus, we get the following:

The Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ (or if $\lim_{n\to\infty} a_n$ DNE) then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

$$\lim_{n\to\infty} \frac{n}{n+1} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{1+0} = 1$$

Since $\lim_{n\to\infty} \frac{n}{n+1} \neq 0$, $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the divergence test.

Ex: $\sum_{n=1}^{\infty} \sec\left(\frac{1}{n}\right) = \sec(1) + \sec\left(\frac{1}{2}\right) + \sec\left(\frac{1}{3}\right) + \cdots$

$$\lim_{n\to\infty} \sec\left(\frac{1}{n}\right) = \sec\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sec(0) = 1 \quad (\neq 0)$$

Thus, $\sum_{n=1}^{\infty} \sec\left(\frac{1}{n}\right)$ diverges by the divergence test.

Ex: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

In this case we have $\lim_{n\to\infty} \frac{1}{n \cdot \ln n} = 0$. What can we conclude from this? NOTHING!

Important Remark:

The divergence test gives no information if $\lim_{n\to\infty} a_n = 0$.

The series could converge or diverge!

Ex: Both $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ satisfy $\lim_{n\to\infty} a_n = 0$,

yet $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.