Applications of Differential Equations

Differential equations can be used to model all sorts of continuously changing or evolving processes, including

- heat transfer
- propagation of light / sound / water waves
- vibrations of a guitar string
- movement of celestial bodies

In this section, we'll explore three applications of the first-order DEs from §15.2 and §15.3.

① Newton's Law of Heating / Cooling

This law states :

> The temperature of an object changes at a rate proportional to the difference between the temperature of the object and the temperature of its surroundings.

We can describe this mathematically using a DE:

temperature of object at time t

$$\boxed{\frac{dT}{dt} = -k\left(T(t) - T_s\right)}$$

proportionality constant
(depends on object's material properties)

temperature of surroundings (constant)

We can solve this (separable) DE to determine the temperature function $T(t)$.

$$\frac{dT}{dt} = -k\left(T - T_s\right) \Rightarrow \int \frac{1}{T - T_s}\, dT = \int -k\, dt$$

$$\Rightarrow \ln|T - T_s| = -kt + C$$

$$\Rightarrow |T - T_s| = e^{-kt + C} = e^{C} e^{-kt}$$

$$\Rightarrow T - T_s = \underbrace{\pm e^{C}}_{= A} e^{-kt}$$

$$\Rightarrow \boxed{T(t) = T_s + A e^{-kt}}$$

Remarks:

(i) We can solve for A and k given initial conditions

(ii) As $t \longrightarrow \infty$, we have

$$T(t) = T_s + A\cancel{e^{-kt}}^{\,0} \xrightarrow{t \to \infty} T_s$$

That is, the temperature approaches the temperature of the surroundings, as we would expect!

Ex: A pot of curry is heated to $45^\circ C$ and is placed in a $25^\circ C$ room to cool. After $t = 1$ hour, the curry has cooled to $35^\circ C$. What will the curry's temperature be after $t = 2$ hours in the room?

Solution: The curry cools according to the DE

$$\frac{dT}{dt} = -k(T - 25).$$

From our earlier work, we know

$$T(t) = 25 + Ae^{-kt}$$

Using $T(0) = 45$, we have

$$45 = 25 + Ae^0 = 25 + A \Rightarrow A = 20$$

Using $T(1) = 35$, we have

$$35 = 25 + 20e^{-k(1)} \Rightarrow 10 = 20e^{-k}$$

$$\Rightarrow e^{-k} = 1/2$$

$$\Rightarrow k = -\ln(1/2)$$

Thus, $T(t) = 25 + 20e^{\ln(1/2)t}$

$$= 25 + 20\left[e^{\ln(1/2)}\right]^t = 25 + 20 \cdot \left(\frac{1}{2}\right)^t$$

The temperature at $t = 2$ hours is then

$$T(2) = 25 + 20\left(\frac{1}{2}\right)^2 = 25 + 20\left(\frac{1}{4}\right) = \boxed{30^\circ C}$$

② Population Growth

We'll study two models: natural / exponential growth; logistic growth

I. Natural / Exponential Growth

In this model,

> population changes at a rate that is proportional to the size of the population at time t.

As a differential equation, this is

$$\boxed{\frac{dP}{dt} = kP(t)}$$

population at time t

proportionality constant, depends on birth / death rates.

The general solution is $P(t) = Ce^{kt}$ (exercise).

Moreover, $P(0) = Ce^{k(0)} = C$, so C is the initial population.

Summary: The solution to the IVP

$$\frac{dP}{dt} = kP, \quad P(0) = P_0 \quad \text{is} \quad \boxed{P(t) = P_0 e^{kt}}$$

Ex: A population of rabbits grows exponentially beginning with 2 rabbits. After 1 year, there are 20 rabbits. How many rabbits will there be after 100 years?

Solution: From above, $P(t) = 2e^{kt}$. Using $P(1) = 20$, we have

$$20 = 2e^{k(1)} \Rightarrow k = \ln\left(\frac{20}{2}\right) = \ln(10)$$

Hence

$$P(t) = 2e^{\ln(10)t} = 2 \cdot 10^t.$$

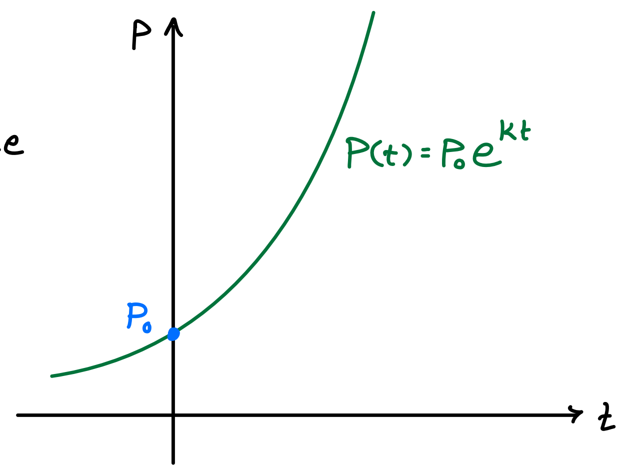
After $t = 100$ years, there will be

$$\boxed{P(100) = 2 \cdot 10^{100} \text{ rabbits}}$$

More rabbits than atoms in the universe!

The previous example shows that the exponential growth model isn't always realistic.

This model predicts that the population will grow endlessly!



But in reality, populations cannot sustain beyond a certain point, known as the carrying capacity.

Our next model accounts for this!

II. Logistic Growth

Let M = carrying capacity. In this model, population changes according to the following DE:

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

Some interesting features:

(i) If $P \ll M$, then $\frac{P}{M} \approx 0$, and hence

$\frac{dP}{dt} \approx kP$ (exponential growth)

(ii) If $P \approx M$, then $\frac{P}{M} \approx 1$ and hence $\frac{dP}{dt} \approx 0$

(slow or no growth since P is near capacity)

(iii) If $P > M$, then $\frac{dP}{dt} < 0$ (population is over capacity and declines)

Let's solve this (separable) DE for P(t)!

$$\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right) \Rightarrow \int \frac{1}{P\left(1-\frac{P}{M}\right)}\,dP = \int k\,dt$$

$$\Rightarrow \int \frac{1}{P\left(\frac{M-P}{M}\right)}\,dP = kt+C$$

$$\Rightarrow \int \frac{M}{P(M-P)}\,dP = kt+C$$

[Using partial fractions, $\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$]

$$\Rightarrow \int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = kt+C$$

$$\Rightarrow \ln|P| - \ln|M-P| = kt+C$$

$$\Rightarrow \ln\left|\frac{P}{M-P}\right| = kt+C$$

$$\Rightarrow \left|\frac{P}{M-P}\right| = e^{kt+C} = e^{kt}e^{C}$$

$$\Rightarrow \frac{P}{M-P} = \pm e^{C}e^{kt}$$

$$\Rightarrow \frac{M-P}{P} = \frac{1}{\pm e^{C}e^{kt}} = \underbrace{\pm e^{-C}}_{=A}e^{-kt}$$

$$\Rightarrow \frac{M}{P} - 1 = Ae^{-kt}$$

$$\Rightarrow \boxed{P(t) = \frac{M}{1+Ae^{-kt}}}$$

In fact, we can determine the constant A. Indeed,

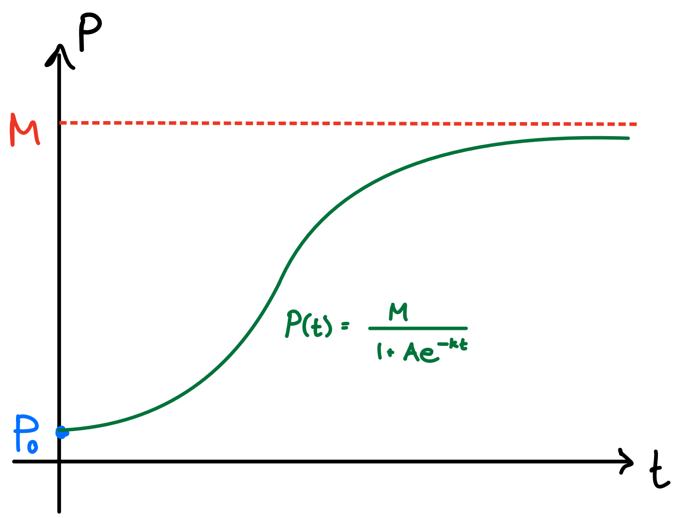
if $P(0) = P_0$ = initial population, then

$$P_0 = \frac{M}{1+Ae^0} = \frac{M}{1+A} \Rightarrow P_0 + AP_0 = M \Rightarrow \boxed{A = \frac{M-P_0}{P_0}}$$

Summary: The solution to the IVP

$$\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right), \quad P(0) = P_0 \quad \text{is}$$

$$\boxed{P(t) = \frac{M}{1+Ae^{-kt}}, \quad \text{where} \quad A = \frac{M-P_0}{P_0}.}$$



Ex: A population of geese starts with 100 geese and grows logistically with carrying capacity 1500. Suppose there are 150 geese after 1 year.

(a) Find the population function, $P(t)$.

(b) After how many years will the population reach 500?

Solution: (a) Population changes according to the DE

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{1500}\right).$$

From above, the solution is $P(t) = \dfrac{1500}{1 + Ae^{-kt}}$ where

$$A = \frac{M - P_0}{P_0} = \frac{1500 - 100}{100} = 14$$

Thus,

$$P(t) = \frac{1500}{1 + 14e^{-kt}}.$$

We can use $P(1) = 150$ to find k.

$$150 = \frac{1500}{1+14e^{-k\cdot 1}} \quad \Rightarrow \quad 1+14e^{-k} = \frac{1500}{150} = 10$$

$$\Rightarrow \quad e^{-k} = 9/14$$

$$\Rightarrow \quad k = -\ln(9/14)$$

Hence, $\boxed{P(t) = \frac{1500}{1+14e^{\ln(9/14)t}} = \frac{1500}{1+14\cdot\left(\frac{9}{14}\right)^t}}$

(b) We need to find t such that $P(t) = 500$.

$$500 = \frac{1500}{1+14\left(\frac{9}{14}\right)^t} \quad \Rightarrow \quad 1+14\left(\frac{9}{14}\right)^t = \frac{1500}{500} = 3$$

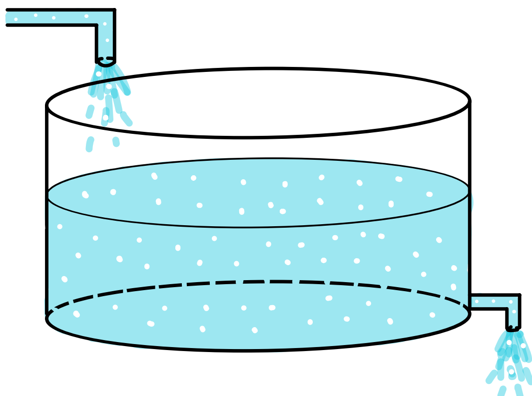
$$\Rightarrow \left(\frac{9}{14}\right)^t = \frac{2}{14} = \frac{1}{7}$$

$$\therefore \quad t = \log_{9/14}\left(\tfrac{1}{7}\right) = \frac{\ln(1/7)}{\ln(9/14)} \approx 4.4$$

The population will reach 500 after $\boxed{\approx 4.4 \text{ years}}$

③ Mixing Problems

Ex: A tank contains 1000L of salt water at a concentration of 0.5 kg/L. Salt water at concentration 0.2 kg/L flows into the tank at a rate of 10 L/min, is thoroughly mixed, and then flows out at the same rate.



Determine the amount of salt in the tank at time t.

Solution: Let $A(t)$ denote the amount of salt in the tank at time t. We have

$$\frac{dA}{dt} = \begin{matrix}\text{rate of} \\ \text{salt in}\end{matrix} - \begin{matrix}\text{rate of} \\ \text{salt out}\end{matrix}$$

Here,

$$\begin{matrix}\text{rate of} \\ \text{salt in}\end{matrix} = \Big(\underset{\text{concentration}}{0.2\ \mathrm{kg/L}}\Big)\Big(\underset{\text{flow}}{10\ \mathrm{L/min}}\Big) = 2\ \mathrm{kg/min}$$

For the rate of salt out, note that as the new solution is mixed into the tank, we have

$$\begin{matrix}\text{concentration} \\ \text{at time } t\end{matrix} = \frac{\text{amount of salt at time } t}{\text{volume at time } t}$$

$$= \frac{A(t)}{1000} \longleftarrow \text{Volume is constant!}$$

Hence,

$$\begin{matrix}\text{rate of} \\ \text{salt out}\end{matrix} = \Big(\underset{\text{concentration}}{\frac{A}{1000}\ \mathrm{kg/L}}\Big)\Big(\underset{\text{flow}}{10\ \mathrm{L/min}}\Big) = \frac{A}{100}\ \mathrm{kg/min}$$

We therefore solve the (separable) DE

$$\frac{dA}{dt} = 2 - \frac{A}{100} = \frac{-1}{100}(A-200).$$

We get

$$\int \frac{1}{A-200}\, dA = \int \frac{-1}{100}\, dt$$

$$\Rightarrow \quad \ln|A-200| = -\frac{t}{100} + C$$

$$\Rightarrow \quad |A-200| = e^{-t/100 + C} = e^{-t/100} e^{C}$$

$$\Rightarrow \quad A = 200 \underbrace{\pm e^{C}}_{=B} e^{-t/100}$$

$$\Rightarrow \quad A = 200 + Be^{-t/100}$$

Initially, the tank contains $0.5\ \mathrm{kg}/\mathrm{L} \cdot 1000\ \mathrm{L} = 500\ \mathrm{kg}$ of salt, hence $A(0) = 500$. This gives

$$500 = 200 + Be^{0} = 200 + B \quad \Rightarrow \quad \underline{B = 300}$$

$$\therefore \boxed{A(t) = 200 + 300e^{-t/100}}$$