③ The Comparison Tests

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n^5+2}$ converge or diverge?

This would be easy if the "+2" weren't present, since $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a convergent p-series ($p=5>1$).

We can, however, say that

$$\frac{1}{n^5+2} \leq \frac{1}{n^5} \quad \text{for all } n.$$

So the terms of our series $\sum_{n=1}^{\infty} \frac{1}{n^5+2}$ are even smaller than those of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^5}$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^5+2}$ must also converge!

The above argument is the idea behind...

The Comparison Test

Suppose that $0 \leq a_n \leq b_n$ for all n sufficiently large.

(i) If $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Note: If $\sum a_n$ converges or $\sum b_n$ diverges, the comparison test gives no information!

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

We have $0 \leq \dfrac{2^n}{3^n+1} \leq \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$ and $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series ($|r| = 2/3 < 1$).

Consequently, $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$ converges by comparison.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)+1}{n^8}$

Note that since $0 \leq \cos^2(n) \leq 1$, we have

$$\frac{\cos^2(n)+1}{n^8} \leq \frac{1+1}{n^8} = \frac{2}{n^8}.$$

The series $\sum_{n=1}^{\infty} \frac{2}{n^8} = 2\sum_{n=1}^{\infty} \frac{1}{n^8}$ converges, as it

is 2 times a convergent p-series. Therefore,

$\sum_{n=1}^{\infty} \frac{\cos^2(n)+1}{n^8}$ converges by comparison.

Ex: $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+n+1}}{n^2}$

We have $\frac{\sqrt{n^3+n+1}}{n^2} \geqslant \frac{\sqrt{n^3}}{n^2} = \frac{n^{3/2}}{n^2} = \frac{1}{\sqrt{n}}$, and

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series (as $p = \frac{1}{2} \leq 1$).

Thus, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+n+1}}{n^2}$ also diverges by comparison.

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge or diverge?

Idea: Try removing the "-1" in the denominator.

We have $\frac{1}{2^n-1} \geqslant \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ for all n and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \text{ is a convergent geometric series } \ldots$$

... but this isn't helpful!

Recall: Being larger than a convergent series or smaller than a divergent series tells us nothing!

We still guess that $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges, as $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and $\frac{1}{2^n-1} \approx \frac{1}{2^n}$ when n is large.

To make this precise, we'll need a new test!

The Limit Comparison Test (LCT)

Let $\sum a_n$ and $\sum b_n$ be series of positive terms, and let

$$\boxed{L = \lim_{n \to \infty} \frac{a_n}{b_n}}$$

If L exists and $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Back to the example of $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$:

Let's try the LCT with $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$

We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}\left(1-\underbrace{\frac{1}{2^n}}_{\to 0}\right)} = \frac{1}{1-0} = 1.$$

Since L exists and $0 < L < \infty$, the LCT implies that $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ either both converge or both diverge. Thus, since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ must converge too!

Ex: Does $\sum_{n=1}^{\infty} \frac{n}{n^2+6}$ converge or diverge?

Previously we used the integral test to show this is divergent... but that took a lot of work!

Solution: Let's try the LCT with $a_n = \frac{n}{n^2+6}$ and

$b_n = \frac{n}{n^2} = \frac{1}{n}$.

Tip: Use the most dominant term in the numerator and denominator to define b_n!

We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n}{n^2+6}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+6}$$

$$\overset{LH}{=} \lim_{n\to\infty} \frac{2n}{2n} = 1 \in (0,\infty).$$

By the LCT, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it's the harmonic series!), $\sum_{n=1}^{\infty} \frac{n}{n^2+6}$ must also diverge.

Ex: Does $\sum_{n=1}^{\infty} \frac{\sqrt{n^6+4}}{2n^5+1}$ converge or diverge?

Solution: Use the LCT with $a_n = \frac{\sqrt{n^6+4}}{2n^5+1}$ and

$b_n = \frac{\sqrt{n^6}}{n^5} = \frac{n^3}{n^5} = \frac{1}{n^2}$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^6+4}}{2n^5+1}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2\sqrt{n^6+4}}{2n^5+1}$$

$$= \lim_{n\to\infty} \frac{n^2\sqrt{n^6}\sqrt{1+4/n^6}}{2n^5+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^5}\sqrt{1+4/n^6}}{\cancel{n^5}(2+1/n^5)} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2} \in (0,\infty).$$

Thus, by the LCT, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$, $\sum_{n=1}^{\infty} \frac{\sqrt{n^6+4}}{2n^5+1}$ must converge too!