§9.4 – Calculus with Polar Curves

A polar curve $r = f(\theta)$ can be thought of as a parametric curve with

$$\left.\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta \\ y &= r\sin\theta = f(\theta)\sin\theta \end{aligned}\right\} \text{parametric equations with parameter } \theta \text{!}$$

Thus, we can use our knowledge of parametric curves to calculate slopes and arc lengths of polar curves!

Derivatives of Polar Curves

The slope of the tangent line to a polar curve $r = f(\theta)$ is

$$\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad \text{where} \quad \begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}}$$

Same as $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ from §9.1!

Ex: Find the slope of the tangent line to $r = \cos(2\theta)$

at $\theta = \pi/4$.

Solution: We have $x = r\cos\theta = \cos(2\theta)\cos\theta$, hence

$y = r\sin\theta = \cos(2\theta)\sin\theta$

product rule!

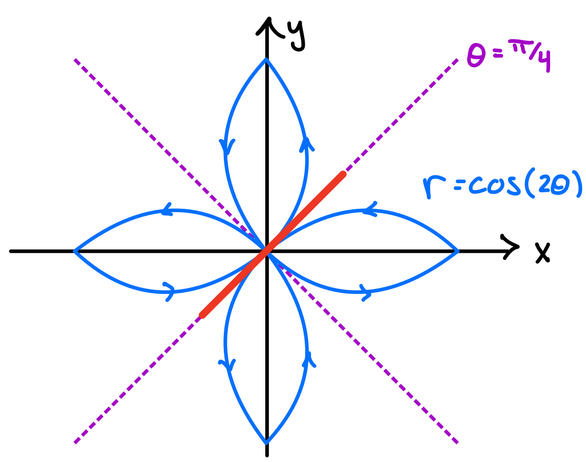
$$\text{Slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin(2\theta)\cdot\sin\theta + \cos(2\theta)\cos\theta}{-2\sin(2\theta)\cdot\cos\theta - \cos(2\theta)\sin\theta}$$

At $\theta = \pi/4$, we get

$$\text{Slope} = \frac{-2\sin(\pi/2)\sin(\pi/4) + \underbrace{\cancel{\cos(\pi/2)}}_{=0}\cos(\pi/4)}{-2\sin(\pi/2)\cos(\pi/4) - \underbrace{\cancel{\cos(\pi/2)}}_{=0}\sin(\pi/4)}$$

$$= \frac{-2\cdot 1\cdot(\sqrt{2}/2) + 0}{-2\cdot 1\cdot(\sqrt{2}/2) - 0}$$

$$= \boxed{1}$$



Arc Length of a Polar Curve

The arc length of a polar curve $r = f(\theta)$ for $\theta \in [a,b]$

is given by $\displaystyle\int_{\theta=a}^{\theta=b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

(Same as $\displaystyle\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ from §9.1 !)

With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, this becomes . . .

product rule!

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left[f'(\theta)\cos\theta - f(\theta)\sin\theta\right]^2 + \left[f'(\theta)\sin\theta + f(\theta)\cos\theta\right]^2$$

$$= [f'(\theta)]^2\cos^2\theta - \cancel{2f(\theta)\sin\theta f'(\theta)\cos\theta} + [f(\theta)]^2\sin^2\theta$$
$$+ [f'(\theta)]^2\sin^2\theta + \cancel{2f(\theta)\cos\theta f'(\theta)\sin\theta} + [f(\theta)]^2\cos^2\theta$$

$= [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) = [f'(\theta)]^2$ $\qquad$ $= [f(\theta)]^2(\sin^2\theta + \cos^2\theta) = [f(\theta)]^2$

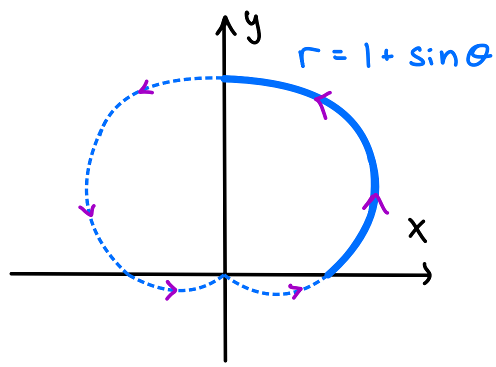
$$= [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

Hence, the arc length of a polar curve $r = f(\theta)$ for $\theta \in [a,b]$ is

$$\boxed{\int_{\theta=a}^{\theta=b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

<u>Ex:</u> Find the length of the cardioid $r = 1 + \sin\theta$ in the first quadrant of the xy-plane.

<u>Solution:</u>



$$\text{Arc Length} = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{1 + 2\sin\theta + (\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2 + 2\sin\theta} \cdot \frac{\sqrt{2 - 2\sin\theta}}{\sqrt{2 - 2\sin\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta$$

Let $u = 2-2\sin\theta$

$du = -2\cos\theta \, d\theta$

$$= \int_2^0 \frac{-1}{\sqrt{u}}\, du$$

Improper due to asymptote at $u=0$!

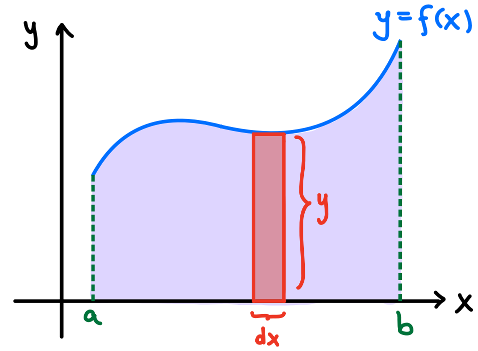
$$= \lim_{t\to 0^+} \int_2^t -u^{-1/2}\, du$$

$$= \lim_{t\to 0^+} -\left[\frac{u^{1/2}}{1/2}\right]_2^t$$

$$= \lim_{t\to 0^+} -2\left[\sqrt{t}-\sqrt{2}\right] = \boxed{2\sqrt{2}}$$

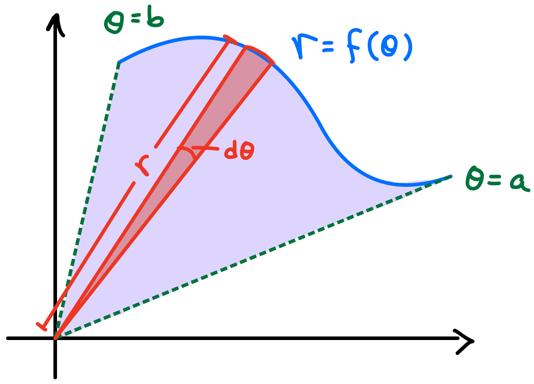
Area Enclosed by a Polar Curve

In Cartesian coordinates, we compute area as $\displaystyle\int_{x=a}^{x=b} y\, dx$



Can we do something similar to find the area enclosed by a polar curve $r = f(\theta)$ for $\theta \in [a,b]$?

We'll divide the region into sectors with radius r and and small angle $d\theta$.



$$\text{Area of sector} = \pi \cdot \text{radius}^2 \cdot \left(\frac{\text{angle}}{2\pi}\right)$$

$$= \pi r^2 \cdot \frac{d\theta}{2\pi}$$

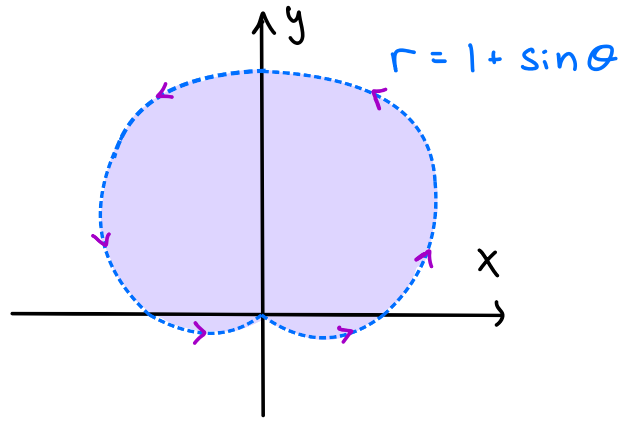
$$= \frac{1}{2} r^2 \, d\theta$$

Hence, the area enclosed by $r = f(\theta)$ for $\theta \in [a,b]$ is

$$\boxed{\int_{\theta=a}^{\theta=b} \frac{1}{2} r^2 \, d\theta}$$

Ex: Calculate the area enclosed by the polar curve $r = 1 + \sin\theta$ for $\theta \in [0, 2\pi]$.

Solution:



$$\text{Area} = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left(1+\sin\theta\right)^2 d\theta$$

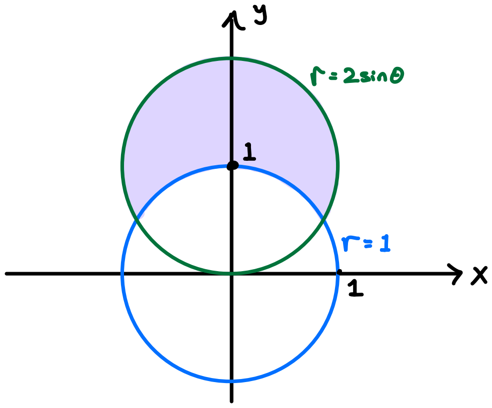
$$= \frac{1}{2} \int_0^{2\pi} \left(1+2\sin\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1+2\sin\theta + \frac{1}{2}\left(1-\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2} \left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{2} \cdot \frac{\sin(2\theta)}{2} \right]_0^{2\pi} = \boxed{\frac{3\pi}{2}}$$

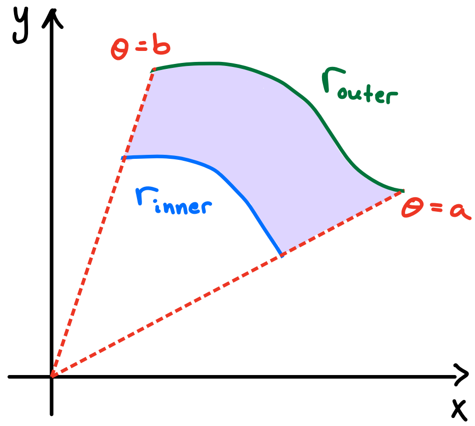
Ex: Set up — but do not evaluate — an integral that represents the area inside $r = 2\sin\theta$ and outside $r = 1$, as shown below.



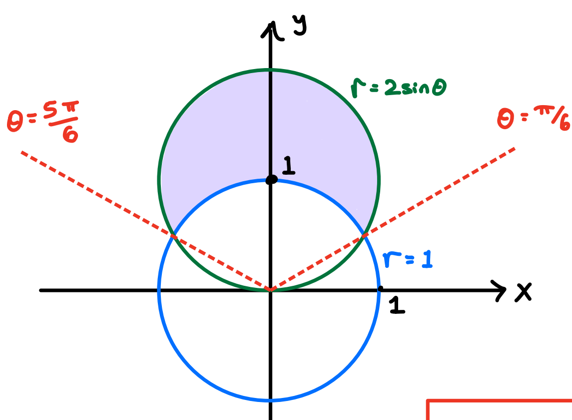
[Exercise: Sketch these yourself!]

Note: The area between two polar curves for $\theta \in [a,b]$ can be computed as

$$\boxed{\int_{\theta=a}^{\theta=b} \frac{1}{2}\left[(r_{outer})^2 - (r_{inner})^2\right] d\theta}$$



Solution: Here, $r_{outer} = 2\sin\theta$ and $r_{inner} = 1$.



The curves $r=2\sin\theta$ and $r=1$ intersect when

$$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \underline{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$$

$$\boxed{\therefore \text{Area} = \int_{\theta=\pi/6}^{\theta=5\pi/6} \frac{1}{2}\left[(2\sin\theta)^2 - (1)^2\right] d\theta}$$