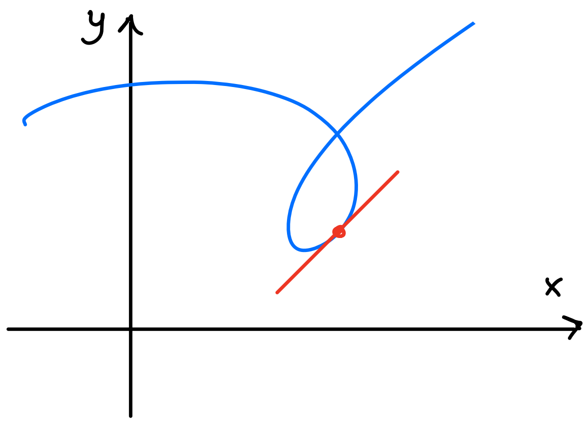
§9.1 (continued) — Calculus with Parametric Curves

Derivatives & Tangents

The slope of the tangent line to a parametric curve is calculated as

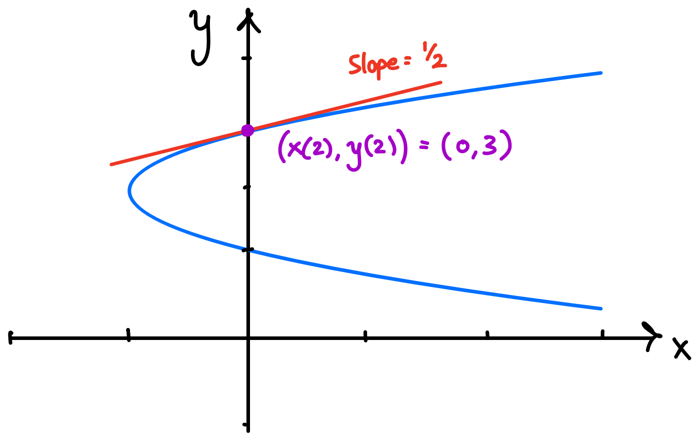
$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$



Ex: Consider $x = t^2 - 2t$, $y = t + 1$. Find the slope of the tangent line when $t = 2$.

Solution: We have $\frac{dx}{dt} = 2t - 2$ and $\frac{dy}{dt} = 1$, hence

$$\text{slope} = \left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{dy/dt}{dx/dt}\right|_{t=2} = \left.\frac{1}{2t-2}\right|_{t=2} = \boxed{\frac{1}{2}}$$



Special Cases:

(i) If $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$ at $t = t_0$, the curve has a horizontal tangent line at $t = t_0$.

(ii) If $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$ at $t = t_0$, the curve has a vertical tangent line at $t = t_0$.

(iii) If $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$ at $t = t_0$, we must examine $\displaystyle\lim_{t \to t_0} \frac{dy/dt}{dx/dt}$ to determine the slope at t_0.

Ex: Find all points (x,y) where the tangent line is vertical or horizontal, then sketch the graph.

(a) $x = \sin(\pi t)$, $y = 8t^3$, $t \in [-1,1]$.

Solution: (a) First, $\dfrac{dx}{dt} = \pi\cos(\pi t)$, $\dfrac{dy}{dt} = 24t^2$

Horizontal tangents?

We need $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$.

$\dfrac{dy}{dt} = 24t^2 = 0 \Rightarrow t = 0$

(and $\dfrac{dx}{dt} = \pi\cos(\pi t) \neq 0$ here!)

Vertical tangents?

We need $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$.

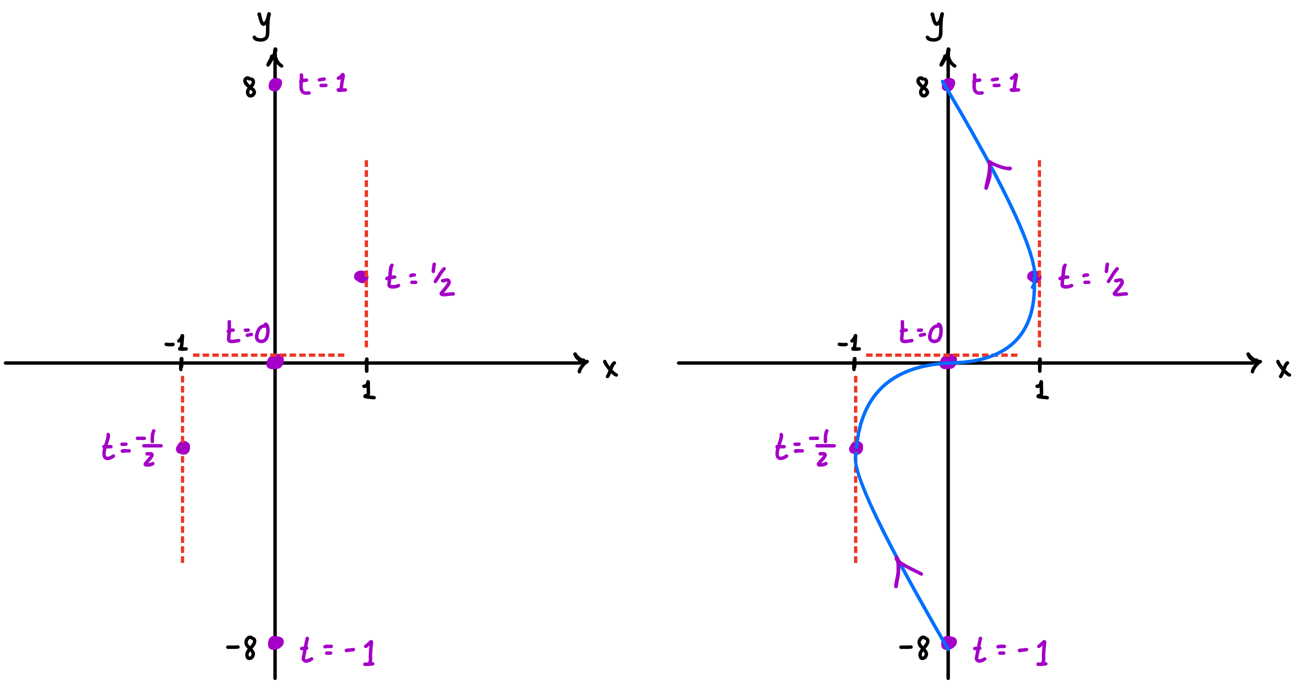
$\dfrac{dx}{dt} = \pi\cos(\pi t) = 0 \Rightarrow t = \pm\frac{1}{2}$

(and $\dfrac{dy}{dt} = 24t^2 \neq 0$ here!)

Summary:

t	0	$1/2$	$-1/2$
$(x(t), y(t))$	$(0,0)$	$(1,1)$	$(-1,-1)$
	Horizontal Tangent	Vertical Tangents	

To sketch the graph, let's plot the starting point $(x(-1), y(-1)) = (0,-8)$, the ending point $(x(1), y(1)) = (0,8)$, as well as the above points and their tangent lines.



(b) $x = \cos(t)$, $y = \sin(2t)$, $t \in [0, 2\pi]$.

Solution: We have $\frac{dx}{dt} = -\sin(t)$ and $\frac{dy}{dt} = 2\cos(2t)$.

Horizontal tangents? Need $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\Rightarrow \frac{dy}{dt} = 2\cos(2t) = 0$

$\Rightarrow 2t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Rightarrow t = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \ldots$ Also $\frac{dx}{dt} = -\sin(t) \neq 0$ here!

$\Rightarrow t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ since $t \in [0, 2\pi]$

Vertical tangents? Need $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

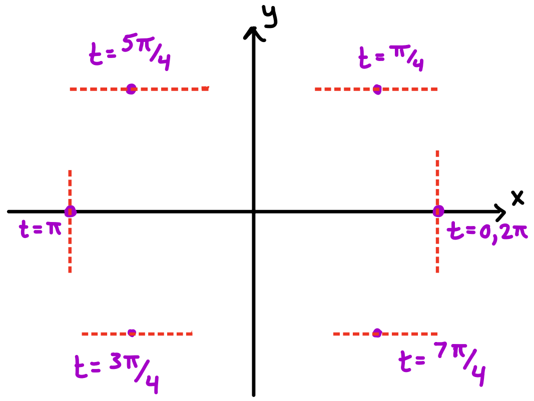
$\Rightarrow -\sin(t) = 0$

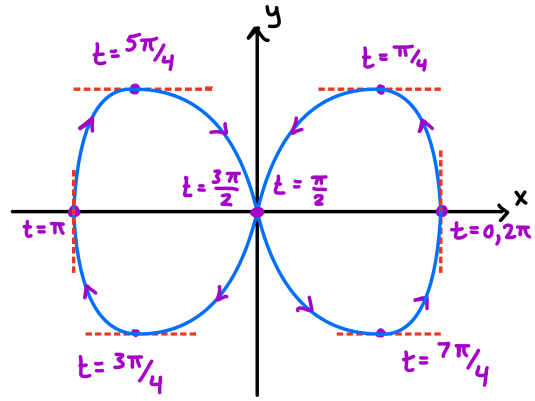
$\Rightarrow t = 0, \pi, 2\pi \longleftarrow$ Also $\frac{dy}{dt} = 2\cos(2t) \neq 0$ here!

Summary:

t	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$	$0, 2\pi$	π
$(x(t), y(t))$	$(\sqrt{2}/2, 1)$	$(-\sqrt{2}/2, -1)$	$(-\sqrt{2}/2, 1)$	$(\sqrt{2}/2, -1)$	$(1, 0)$	$(-1, 0)$
	Horizontal tangents				Vertical tangents	

We plot these points below and sketch the graph:





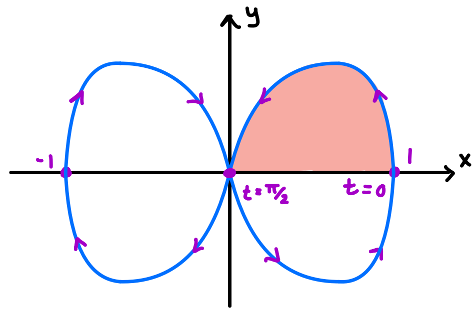
Area Enclosed by a Parametric Curve

Q: What is the total area enclosed by the parametric curve

$x = \cos(t)$, $y = \sin(2t)$, $t \in [0, 2\pi]$ from the previous example?

A: Due to symmetry,

Total Area = 4 · (Shaded Area)



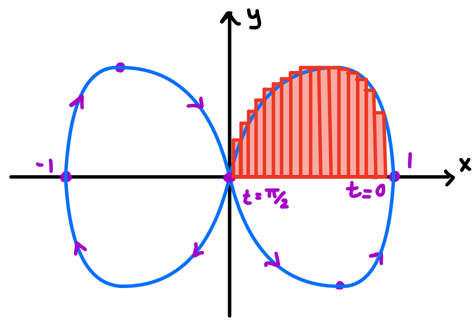
$$= 4 \int_{x=0}^{x=1} y \, dx$$

$x = \cos t = 1 \Rightarrow t = 0$

$x = \cos t$
$dx = -\sin t \, dt$

$y = \sin(2t)$

$x = \cos t = 0 \Rightarrow t = \pi/2$



$$= 4\int_{t=\pi/2}^{t=0} \underbrace{\sin(2t)}_{=2\sin t\cos t}\cdot(-\sin t)\,dt$$

$$= -4\int_{t=\pi/2}^{t=0} 2\sin^2 t\cdot\cos t\,dt \qquad \text{Let } u=\sin t,\ du=\cos t\,dt$$

$$= -8\int_{u=1}^{u=0} u^2\,du$$

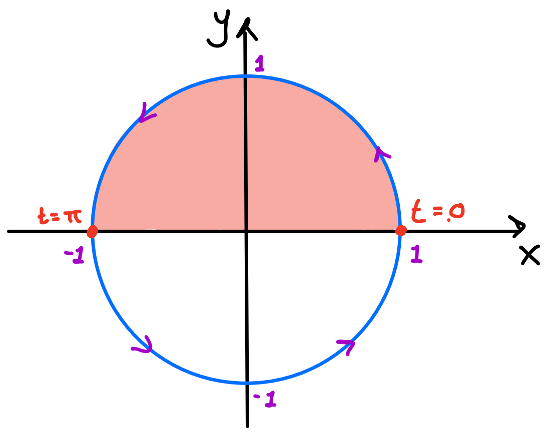
$$= -8\left[\frac{u^3}{3}\right]_1^0 = \boxed{\frac{8}{3}}$$

<u>Ex:</u> Calculate the area enclosed by the parametric curve

$x=\cos t,\ y=\sin t,\ t\in[0,2\pi]$.

<u>Solution:</u> This equation describes the unit circle!

Total Area = 2 · (Shaded Area)



$$= 2\int_{x=-1}^{x=1} y\,dx \qquad x=\cos t,\ dx=-\sin t\,dt$$

$$= 2\int_{t=\pi}^{t=0} \sin t\cdot(-\sin t)\,dt$$

$$= -2\int_{t=\pi}^{t=0} \underbrace{\sin^2 t}_{=\frac{1}{2}(1-\cos 2t)}\, dt$$

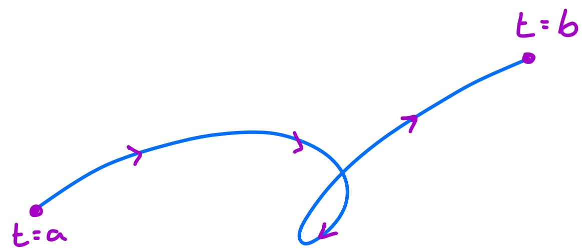
$$= -\int_{t=\pi}^{t=0} (1-\cos 2t)\, dt = \Big[\sin 2t - t\Big]_{\pi}^{0} = \boxed{\pi}$$

Arc Length of a Parametric Curve

__Q:__ Given $x = x(t)$, $y = y(t)$,

what is the length of the

curve from $t = a$ to $t = b$?

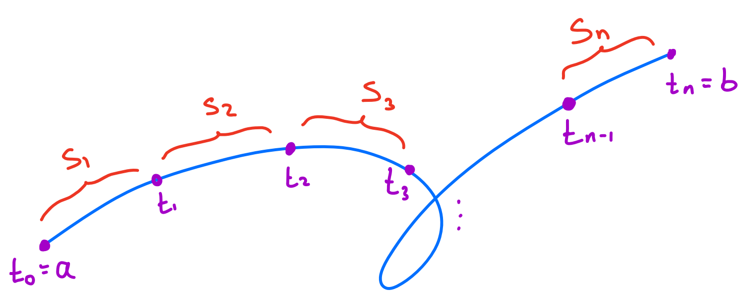


__Idea:__ Let's chop up the curve into smaller pieces and

approximate the length of each piece!

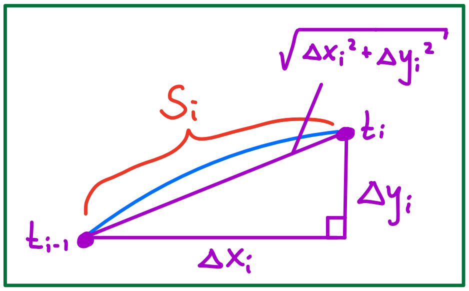
s_i = tiny arc length

$$\text{Arc length} = \sum_{i=1}^{n} s_i$$



$$\approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\ \Delta t$$



Taking a limit as $n \longrightarrow \infty$, we obtain the following:

The arc length of a parametric curve

$$x = x(t),\quad y = y(t),\quad t \in [a, b]$$

is equal to

$$\boxed{\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$$

Ex: Calculate the arc length for each curve below:

(a) $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

Solution: This equation describes the unit circle!

We have $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, and hence

$$\text{Arc length} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{=1}}\, dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

(b) $x = \frac{t^2}{2}$, $y = \frac{(2t+1)^{3/2}}{3}$, $t \in [0,1]$

Solution: $\frac{dx}{dt} = t$, $\frac{dy}{dt} = \frac{\cancel{3/2}\,(2t+1)^{1/2} \cdot \cancel{2}}{\cancel{3}} = \sqrt{2t+1}$, hence

$$\text{Arc length} = \int_0^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{t^2 + \left(\sqrt{2t+1}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{t^2 + 2t + 1}\, dt$$

$$= \int_0^1 \sqrt{(t+1)^2}\, dt = \int_0^1 (t+1)\, dt = \left[\frac{t^2}{2} + t\right]_0^1 = \boxed{\frac{3}{2}}$$